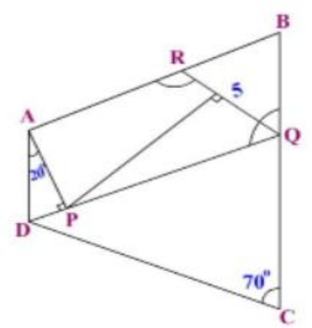
ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC – 2023 November – PRIMARY – FINAL

Instructions:

1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before starting to answer, fill in the **FACE SLIP** completely.
5. Your 'rough work' should be done in the answer sheet itself.
6. Maximum time allowed is **THREE hours**.

Question 1:

ABCD is an isosceles trapezium as shown in the figure, in which AB = DC, ∠DAP = 20°, DP is perpendicular to AP, ∠C = 70°, QR is the bisector of ∠BQD and PS ⊥ QR. Calculate ∠SPQ and ∠SRA. Justify each of the steps in calculation.



Solution:

In the figure,

∠ABC = 70º (Base of isosceles trapezium are equal)

Now, Let ∠BQR be x,

Then, ∠SQP = x

And ∠SPQ = 90º — x

Also, ∠ APQ = 90º

In quadrilateral APQR,

∠APQ + ∠PQR + ∠QRA + ∠RAP = 360º

90º + x + ∠QRA + 90º = 360º

∠QRA = 180º — x ………… Eq(1)

In Triangle RBQ,

∠QRB + ∠RBQ + ∠BQR = 180º

∠QRB + 70 + x = 180º

∠QRB = 110 — x ………… Eq(2)

Now, from Eq(1) and Eq(2)
Since ARB is a straight line,
ARQ + ∠QRB = 180°
180° — x + 110 – x =180°
2x = 180°
x = 55°

Hence, ∠SPQ = 90° — x = 90° — 55° = 35°
And ∠SRA = 180° – x = 180° – 55° = 125°

Question 2:

Ramanujan is a sixth-grade student. His mathematics teacher gave a problem sheet in maths as home task for the Puja holidays. Ramanujan wants to complete it in 4 days and wants to enjoy the holidays for the remaining 6 days.
On the first day, he worked out one-fifth the number of problems plus 12 more problems.
On the second day, he worked out one-fourth the remaining problems plus 15 more problems.
On the third day, he solved one-third of the remaining problems plus 20 more problems.
The fourth day, he worked out successfully the remaining 60 problems and completed the work.
Find the total number of problems given by the teacher and the number of problems solved by Ramanujan on each day.

Solution:

Let the total number of problems be P.
Number of Problems solved by Ramanujan on the first day = $\frac{P}{5} + 12$
Remaining number of problems after the first day

$$P - \left(\frac{P}{5} + 12\right) = \frac{P}{1} - \frac{P}{5} - \frac{12}{1}$$

$$= \frac{5P - P - 60}{5} = \frac{4P - 60}{5}$$

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Number of problems solved by Ramanujan on the second day

$=\frac{1}{4}$ of $\frac{4P-60}{5}$ + 15

$= \frac{4P-60}{20} + 15$

$= \frac{4P - 60 + 300}{20}$

$= \frac{4P + 240}{20}$

Now, let us find the total number of problems solved on first and the second day:

$\frac{P}{5} + 12 + \frac{4P + 240}{20}$

$= \frac{4P + 240 + 4P + 240}{20}$

$= \frac{8P + 480}{20}$

Remaining problems after the second day:

$P - \frac{8P + 480}{20}$

$= \frac{20P - 8P - 480}{20}$

$= \frac{12P - 480}{20}$

Number of problems solved by Ramanujan on the third day:

$\frac{1}{3}$ of $\frac{12P - 480}{20}$ + 20

$= \frac{12P - 480}{60}$ + 20

$= \frac{12P - 480 + 1200}{60}$

$= \frac{12P + 720}{60}$

Now let us find the total number of problems of solved in three days:

Day 1(problem solved) + Day 2(problem solved) + Day 3(problem solved)

$= \frac{8P + 480}{20} + \frac{12P + 720}{60}$

Solving the expression above, we get:

$= \frac{36P + 2160}{60}$

Now let us find the remaining problems after third day:

P — $\frac{36P + 2160}{60}$

$= \frac{60P - 36P + 2160}{60}$

$= \frac{24P - 2160}{60}$

According to the question, the number of problems remaining to be solved after the third day is 60.

Therefore,
$\frac{24P - 2160}{60} = 60$
24P — 2160 = 60×60
24P = 3600 + 2160
24P = 5760
P = 5760/24
P = 240
Therefore, the total number of Problems solved by Ramanujan is 240.

Now let us find the number of Problems solved by Ramanujan on each day respectively.
Number of Problems solved by Ramanujan on the first day = $\frac{P}{5} + 12 = 48 + 12 = 60$
Number of Problems solved by Ramanujan on the second day = $\frac{4P + 240}{20} = 60$
Number of Problems solved by Ramanujan on the third day = $\frac{12P + 720}{60} = 60$

Question 3:

There are 4 cards and, on each card, a whole number is written. All the numbers are different from one another. Two girls of grade six, Deepa and Dilruba play a game.
Deepa takes 3 cards at a time leaving a card behind. She multiplies the numbers and gets an answer. In the same way, again, she leaves one different card and selects the other three and multiplies the numbers.
She got the answers 480, 560, 420 and 336.
Now, Dilruba has to find the numbers in each card. Dilruba worked out and found the correct numbers.
What are they? Work out systematically and find the numbers.

Solution:

Let us assume the cards to be “a” , “b” , “c” and “d” such that a ,b, c and d are distinct whole numbers.
Now Deepa picks 3 cards at a time leaving one behind such that she gets 480, 560, 420 and 336
Now we can write a x b x c = 560
b x c x d = 480

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a x b x d = 420
c x d x a = 336

Now if were to prime factorize these numbers we get,
480 = 2 x 2 x 2 x 2 x 2 x 3 x 5
560 = 2 x 2 x 2 x 2 x 5 x 7
420 = 2 x 2 x 3 x 5 x 7
336 = 2 x 2 x 2 x 2 x 3 x 7

But knowing that c x d x a = 336 , c, d and a cannot be a multiple of 5 as 336 does not end with a 0 or 5. Therefore b is definitely the only multiple of 5

Now using 336 = 2 x 2 x 2 x 2 x 3 x 7 , let us take the factors as 8 , 6 and 7. We can do this as through prime factorization we can get many factor pairs and triplets.

Now c = 8 , d = 6 , a =7 using these numbers in the remaining relations that we have
b x c x d = 480
⇒ b x 8 x 6 = 480
⇒ b x 48 = 480
⇒ b = 10

Therefore, now our whole numbers are 6, 7, 8 and 10 and we can see that these numbers satisfy all the relations mentioned above.

Therefore, the correct solution for the question is 6, 7, 8, 10.

Question 4:

An angle is divided into 3 equal parts by two straight lines; such lines are called trisectors. ABCD is a square. The lines (AP, AS) are trisectors of ∠BAD. Similarly, we have the trisectors (BP, BQ), (CQ, CR) and (DR, DS). Prove that PQRS is a square.

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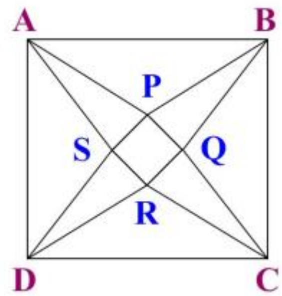
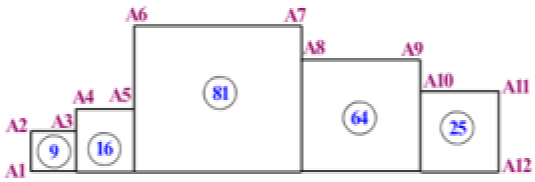


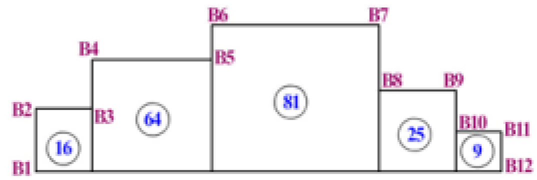
Solution:

Since AP and AS are trisectors then the value of angle PAS will be 30^o
And AP = AS
Similarly, $\angle$ PBQ = 30^o and PB = BQ
Now triangle APS is congruent to triangle BPQ.
$\Rightarrow$ PS = PQ
Similarly, we can prove QR=RS
Now all four sides are equal. So PQRS could be either square or rhombus.
In triangle APS $\angle$ P = $\angle$ S = 75^o
In triangle ASD, $\angle$ S = 120^o
In triangle SDR $\angle$ S = 75^o
Now at point S the total sum of angles should be 360^o.
$\angle$ PSR = 360^o - $\angle$ ASD - $\angle$ DSR - $\angle$ ASP
$\angle$ PSR = 360^o - 120^o - 75^o - 75^o
$\angle$ PSR = 90^o.
Since all four sides are equal and an included angle is 90^o hence it is proved that PQRS is a square.

Question 5:

Five squares of different dimensions are arranged in two ways as shown in the following diagrams. The numbers inside each square represent its area in square units.





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Calculate the perimeter $A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}A_{11}A_{12}A_1$ and the corresponding perimeter of figure 2. Are they same? If they are different, which is greater?

Solution:

In Figure 1,
Area of 1st Square = 9
So, side of 1st square = $\sqrt{9}$ = 3
Similarly, side of 2nd Square = 4
Side of 3rd square = 9
Side of 4th square = 8
Side of 5th square = 5
So perimeter $A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}A_{11}A_{12}A_1$
= 3 + 3 + 1 + 4 + 5 + 9 + 1 + 8 + 3 + 5 + 5 + 29
= 76

Similarly,
In Figure 2,
Perimeter $B_1B_2B_3B_4B_5B_6B_7B_8B_9B_{10}B_{11}B_{12}B_1$
= 4 + 4 + 4 + 8 + 1 + 9 + 4 + 5 + 2 + 3 + 3 + 29
= 76

So, the perimeter for both figures is the same.

Question 6:

i) In a book, a problem on fractions is given as

$$\frac{1}{3\frac{1}{5}} - \frac{2\frac{1}{4}}{9} + \frac{3\frac{5}{8}}{a} + \frac{\frac{4}{7}}{4\frac{4}{7}}$$

The denominator of the third term is not printed. The answer is given to be 2. What is the missing denominator? Let it be *a.*

ii) Simplify: $\dfrac{1}{3-\dfrac{1}{2-\dfrac{1}{(5/7)}}}$. Let the answer be of the form $\frac{p}{q}$ where p,q have no common factors. Let p,q have no common factors. Let b = $\frac{p}{q}$.

iii) Find the value of $(\frac{1}{a^2} + b)$

Solution:

i) Given that,

$$\frac{1}{3\frac{1}{5}} — \frac{\frac{9}{4}}{9} + \frac{3\frac{5}{8}}{a} + \frac{\frac{4}{7}}{4\frac{4}{7}} = 2$$

$$\frac{1}{\frac{16}{5}} — \frac{\frac{9}{4}}{9} + \frac{29}{8a} + \frac{\frac{4}{7}}{\frac{32}{7}} = 2$$

$$\frac{5}{16} — \frac{1}{4} + \frac{29}{8a} + \frac{1}{8} = 2$$

$$\frac{5}{16} — \frac{4}{16} + \frac{58}{16a} + \frac{2}{16} = 2$$

$$\frac{3}{16} + \frac{58}{16a} = 2$$

$$\frac{58}{16a} = 2 — \frac{3}{16}$$

$$\Rightarrow \frac{58}{16a} = \frac{29}{16}$$

$$\Rightarrow \frac{2}{16a} = \frac{1}{16}$$

a = 2

ii) Simplifying,

$$\frac{1}{3 — \frac{1}{2 — \frac{1}{(5/7)}}}$$

$$= \frac{1}{3 — \frac{1}{2 — \frac{7}{(5)}}}$$

$$= \frac{1}{3 — \frac{1}{\frac{10—7}{(5)}}}$$

$$= \frac{1}{3 — \frac{5}{3}}$$

$$= \frac{1}{\frac{9—5}{3}}$$

b = $\frac{3}{4}$

iii)
$(\frac{1}{a^2} + b)$
a = 2, b = $\frac{3}{4}$
= $(\frac{1}{2^2} + \frac{3}{4})$
= $\frac{1+3}{4}$
= 1
$(\frac{1}{a^2} + b)$ = 1

Question 7:

a and b are two integers, find all pairs such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$. Arrive at your result logically.

Solution:

Given: $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$, where a, b are integers

Let's multiply both sides by 2ab to get,
$ab - 2a - 2b = 0$
Add 2^2 on both the sides
Therefore, $ab - 2a - 2b + 2^2 = 2^2$
On simplifying we get as $(a-2)(b-2) = 2^2$
In general : $(a-n)(b-n) = n^2$ (n is integer)
Therefore the divisors of n^2 are $\pm n^2, \pm n, \pm 1$
Also $-n$ is not possible.
So the general solution when n is a prime number is given by

$(a, b) = (n-n^2, n-1), (n-1, n-n^2), (n+1, n+n^2), (n+n, n+n), (n+n^2, n+1)$

Hence for the equation, $(a-2)(b-2) = 2^2$, we get
$(a, b) = (-2, 1), (1, -2), (3, 6), (4, 4)$ and $(6, 3)$

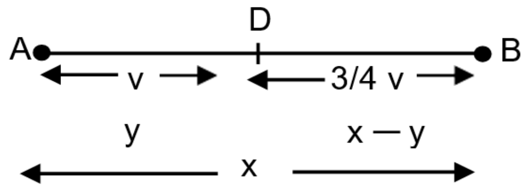
Question 8 :

A train starts from a station A and travels with constant speed up to 100 kms/hr. After some time, there appeared a problem in the engine and so the train proceeds with $\frac{3}{4}$ th of the original speed and arrives at station B, late by 90 min. Had the problem in the engine occurred 60 kms further on , then the train would have reached 15 min sooner. Find the original speed of the train and distance between stations A and B.

Solution:

Let total distance between AB = x
Original speed of the train = v
Total time taken if no problem appeared = t

In first condition, D is the point where problem occurred in the engine,



Here, time taken by train from point A to D

$\Rightarrow t_a = \frac{y}{v}$

Similarly, time taken by train from point D to B

$\Rightarrow t_b = \frac{x-y}{\frac{3}{4}v}$

Now $t_a + t_b = t + \frac{90}{60}$

$\Rightarrow \frac{y}{v} + \frac{4(x-y)}{3v} = \frac{x}{v} + \frac{3}{2}$

$\Rightarrow \frac{y}{v} + \frac{4x}{3v} - \frac{4\,y}{3v} = \frac{x}{v} + \frac{3}{2}$

$\Rightarrow \frac{x}{3v} - \frac{y}{3v} = \frac{3}{2}$

$\Rightarrow x - y = \frac{9}{2}v$ …(1)

Similarly in second condition,

$\Rightarrow \frac{y+60}{v} + \frac{4(x-y-60)}{3v} = \frac{x}{v} + \frac{75}{60}$

$\Rightarrow \frac{y}{v} + \frac{60}{v} + \frac{4x}{3v} - \frac{4y}{3v} - \frac{4\times 60}{3v} = \frac{x}{v} + \frac{75}{60}$

$\Rightarrow \frac{x}{3v} - \frac{y}{3v} - \frac{20}{v} = \frac{5}{4}$

$\Rightarrow \frac{x-y}{3v} - \frac{20}{v} = \frac{5}{4}$

$\Rightarrow \frac{1}{3v} \times \frac{9v}{2} - \frac{20}{v} = \frac{5}{4}$ [Using equation (1), $x - y = \frac{9}{2}v$]

$\Rightarrow \frac{3}{2} - \frac{20}{v} = \frac{5}{4}$

$\Rightarrow \frac{20}{v} = \frac{3}{2} - \frac{5}{4} = \frac{1}{4}$

v = 80 km/hr

Now x - y = $\frac{9}{2}$ x 80 [Putting value of v in equation (1)]

$\Rightarrow$ x - y = 360 km

Hence, original speed of the train is 80 km/hr

With the given information, we cannot find the distance between A and B. (Insufficient information)

BYJU'S Tuition Center NMTC

15th October 2022	Primary Level	Screening test

Instructions:

1. Fill in the Response Sheet with your Name, Class and the Institution through which you appear, in the specified places.
2. Diagrams given are only Visual aids; they are not drawn to scale.
3. You may use separate sheets to do rough work.
4. Use of Electronic gadgets such as Calculator, Mobile Phone or Computer is not permitted.
5. Duration of the Test: 2 pm to 4 pm (2 hours).

Question 1.

In the year 2021, the ratio of A's income to B's income is 5:8. In the next year 2022, if A's income increases by 20% and B's income increases by 15%, what is the ratio of their incomes now?

a) 5:6 b) 7:23 c) 15:23 d) 9:11

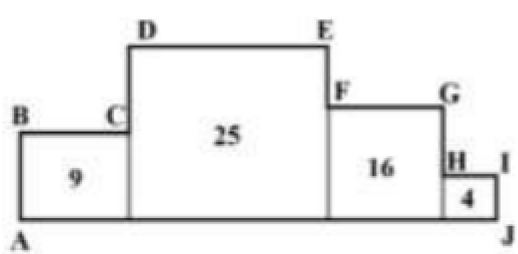
Solution: (c)

Given, A : B = 5 : 8
So, We can take A = 5n and B = 8n
As given in question,
A = 5n + 5n x (20/100)
A = 6n
and
B = 8n + 8n x (15/100)
B = 46n/5
So, A : B = 6n : 46n/5
A : B = 15 : 23

Question 2.

Four squares are placed as shown in the figure. The areas of the squares are marked in the respective squares. The perimeter ABCDEFGHIJA is ...

a) 34 b) 38 c) 36 d) 40



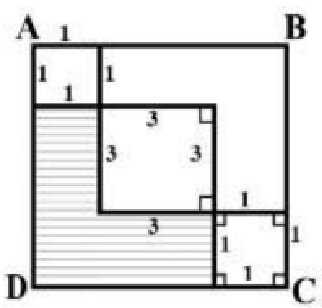
Solution: (b)

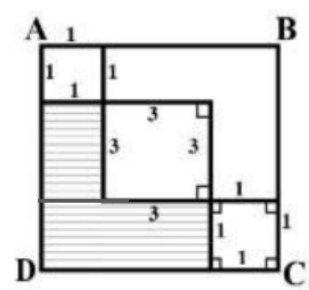
Let's take point K, L and M as shown in the figure
Given, Area of Square ABCM = 9
So, All side of square = 3
Similarly,
All side of square DELM = 5
All side of square FGKL = 4
All side of square HIJK = 2
So, CD = MD - MC = 5 - 3 = 2
Similarly, EF = EL - FL = 5 - 4 = 1
GH = 2
JA = JK + KL + LM + MA
= 2 + 4 + 5 + 3
= 14

Perimeter of ABCDEFGHIJA = AB + BC + CD + DE + EF + FG + GH + HI + IJ + JA
= 3 + 3 + 2 + 5 + 1 + 4 + 2 + 2 + 2 + 14
= 38

Question 3.

In the adjoining figure ABCD is a square and there are two unit squares and a square of side 3 cm. The area of the shaded region (when
given in cm^2) is ...
a) 5 b) 6 c) 7 d) 8





Solution: (c)
Shaded Area is divided into 2 rectangles (as shown in figure)
So, Area of one rectangle = 3 x 1 = 3 cm^2
Area of second rectangle = (3 + 1) x 1
= 4 x 1 = 4 cm^2
Area of shaded region = 3 + 4 = 7 cm^2

Question 4.

The price of an article is reduced by 25%. In order to restore the original price, the new price must be increased by ...

a) 25 % b) 28 % c) 20 % d) 33 1⁄3 %

Solution: (d)

Let,
Old price = x
New price = y
According to question
y = x - (25% of x)
y = x - (25/100 × x)
y = x - x/4
y = 3x/4 ……(1)
Now, in order to restore the original price (x) the new price (y) should be increased by z
x = y + z
4y/3 = y + z ……[∵ from (1)]
z = 4y/3 - y
z = y/3
∴ New price should be increased by One-third means 33 ⅓ %.

Question 5.

The number of three-digit numbers that are multiples of 11 is ...

a) 80 b) 81 c) 79 d) 83

Solution: (b)
Lowest three digit number is 100.
Greatest three digit number is 999.
So the number of Three digit numbers which are divisible by 11= $\frac{999}{11}$ - $\frac{100}{11}$ (take quotient only)

= 90 - 9 = 81

Question 6.

For any natural number n, 2n[3n + {7n (n + 3) - (n + 1) - 2}] is divisible by

a) 7 b) 3 c) 11 d) 13

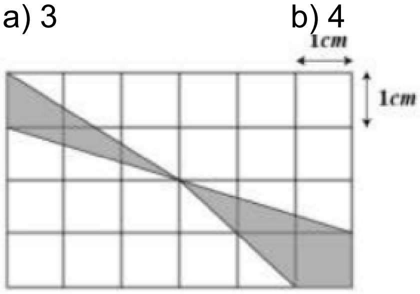
Solution: (b)

Given, 2n[3n + {7n (n + 3) - (n + 1) - 2}]
Let's take n = 1
⇒ 2 x 1 [3 x 1 + {7 x 1 (1 + 3) - (1 + 1) - 2}]
⇒ 2 [3 + {7 (4) - (2) - 2}]
⇒ 2 [3 + { 28 - 4}]
⇒ 2 [3 + 24]
⇒ 2 [27]
⇒ 54

Now, Let's take n = 3
⇒ 2 x 3 [3 x 3 + { 7 x 3 (3 + 3) - (3 + 1) - 2}]
⇒ 6 [9 + { 21 (6) - (4) - 2}]
⇒ 6 [9 + { 126 - 6}]
⇒ 6 [9 + 120]
⇒ 6 [129]
⇒ 774
54 and 774 Both are divisible by 3 only.

Question 7.

In the adjoining figure, the grid consists of Unit squares. The area of the shaded region (in cm^2) is ...
a) 3 b) 4 c) 3.5 d) 4.5



Solution: (b)

So, Area will be
⇒ $\frac{1}{2} \times 1 \times 3 + \frac{1}{2} \times 2 \times 1 + 1 + \frac{1}{2}$
⇒ 4 cm^2

Question 8.

Three digit numbers are formed using the digits 1, 3, 5, 9. The difference between the largest and the smallest numbers thus formed is?

a) 888 b) 798 c) 879 d) 789

Solution: (a)

Largest number = 999
Smallest number = 111
Difference = 999 – 111 = **888**

Question 9.

abc651 is exactly divisible by 5423. Then a + b + c is equal to?

a) 3 b) 6 c) 9 d) None of these

Solution: (d)

11 is a factor of 5423.
So, abc651 is also divisible by 11.
(a + c + 5) – (b + 6 + 1) should be divisible by 11.
So, a + c should be 6 and b should be 4.
Therefore, **a + b + c = 10**

Question 10.

If 12% of a number is 120, then 120% of that number is ?

a) 20 b) 480 c) 120 d) 720

Solution: 1200 (Not in options)

12% of x = 120
x = 120 (100/12)
x = 1000
120% of 1000
= (120/100) 1000
= **1200**

Question 11.

Three natural numbers which are co-prime to one another are such that the product of the first two is 779 and the product of the next two is 1107. The sum of three numbers is?

Solution: 87

Middle number = HCF of 779 and 1107
= 41
First number = 779/41
= 19
Third number= 1107/41
= 27
Sum of the numbers = 41 + 19 + 27 = **87**

Question 12.

The HCF of two natural numbers is 33. The sum of the numbers is 528. The number of such pairs of natural numbers is?

Solution: 4 pairs

Let the numbers be 33a and 33b.
The sum of the numbers is 528.
33a + 33b = 528
33(a+b) = 528
a + b = 528/33
a + b = 16
The combinations are,
a = 1, b = 15
a = 3, b = 13
a = 5, b = 11
a = 7, b = 9
So, there are **4 such pairs.**

Question 13.

If $\left(1\frac{1}{2}\right) \times \left(1\frac{1}{3}\right) \times \left(1\frac{1}{4}\right) \ldots . \times \left(1\frac{1}{n}\right) = \frac{121}{2}, \textit{ then the value of } n \textit{ is?}$

Solution: 120

$\frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times ... \times \frac{n+1}{n} = \frac{121}{2}$

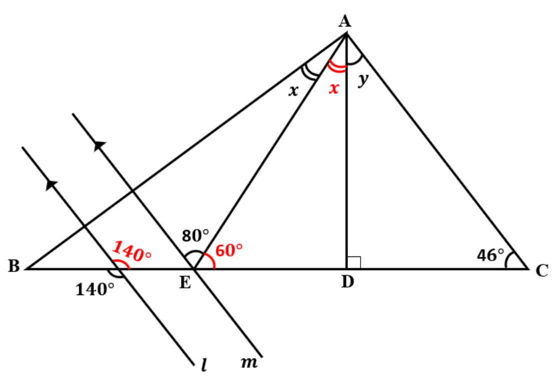
$n + 1 = 121$

$n = 120$

Question 14.

In the adjoining figure, AE is the bisector of angle BAD. The lines l, m are parallel. The degree measure of (x + y) is?

Solution: 74°



y + 90 + 46 = 180

y = 44

60 + x + 90 = 180

x = 30

Therefore, **x + y = 74°**

Question 15.

There are 8 boxes placed in a line. We have 1824 balls to be put in the boxes. Each box has to receive 2 balls more than the previous box. The largest number of balls put in a box is ______________.

Solution: 235

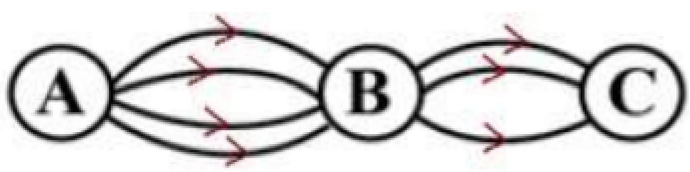
No. of boxes placed in line = 8

No. of balls to be put in boxes = 1824

According to question,
Each box has to receive 2 balls more than the previous box.
So the eq.
1824 = x + (x + 2) + (x + 4) …….. (x + 14)
So, 8x + 56 = 1824
x = 221
We have to find the largest number of balls put in a box = x+ 14
= 221 + 14 = 235

Question 16.

In the adjoining figure, A is your house, B is your friend's house and C is your School. There are 4 paths from A to B and 3 paths from B to C. You want to go to school, picking up your friend. The number of ways you can thus go by different routes is _________.



Solution: 12

As given,
No. of ways to go A to B = 4
No. of ways to go B to C = 3
So, total number of ways by different routes = 4 x 3 = 12

Question 17.

Mrs Sweety had money to buy just 6 Gulabjamoons and 7 Samosas. The sweetshop vendor told her that may also get 8 Gulabjamoons and 4 Samosas, for the same amount. Since Mrs Sweety is a diabetic patient, as per her doctor's advice, she decided not to buy any sweets; so with all the money she had, she bought only Samosas. Thus she got _________ Samosas.

Solution: 16

Let's take gulabjamoon as a and Samosa as b
As given in the question, we can write
6a + 7b = 8a + 4b
So, 2a = 3b

Accordingly we can replace the value of a in 6a + 7b
So, in terms of b, we get 9b + 7b = 16b
That means Sweety can buy 16 samosas in that amount.

Question 18.

In a playing Die, the dots represent values (numbers) from 1 to 6. The opposite 'faces' of a Die add up to 7. In the figure A is a sharing 'vertex' and is given a value 6 (=1×2×3, namely the product of the numbers on faces shared by it). Similar values are given to the other 7 vertices. Then the total value of all the vertices is __________.

Solution: 343

Vertex A = 1 x 2 x 3 = 6
Vertex B = 1 x 2 x 4 = 8
Vertex C = 1 x 5 x 4 = 20
Vertex D = 1 x 5 x 3 = 15
Vertex E = 6 x 2 x 4 = 48
Vertex F = 6 x 2 x 3 = 36
Vertex G = 6 x 5 x 3 = 90
Vertex H = 6 x 4 x 5 = 120

So, The total value of all the vertices = 6 + 8 + 20 + 15 + 48 + 36 + 90 + 120
= 343

Question 19.
A careless Secretary was asked to send 4 letters to 4 different persons. There are 4 envelopes on which separate addresses of the 4 persons were written. The number of ways the Secretary might put wrong letters in all the envelopes is ________.

Solution: 9

As given, Secretary was asked to send 4 letters to 4 different persons

We have to find the number of ways the Secretary might put wrong letters in all the envelopes.
So, Number of ways to distribute the n things to n persons = n!
The concept of de-arrangement
D_n = n! (1/2! - 1/3! + 1/4!)
Here n = 4
D_4 = 4! (1/2! - 1/3! + 1/4!)
= 12 - 4 + 1
= 9

Question 20.

The largest prime factor of the sum of the prime factors of 2022 is _____.

Solution: 19

Prime factors of 2022 = 2, 3, 337
Sum of prime factors = 2 + 3 + 337 = 342
We have to find the largest prime factor of the sum = 342
So, Prime factors of 342 = 2, 3, 19
So, largest prime factor = 19

Regd. Office: Aakash Tower, 8, Pusa Road, New Delhi-110005, Ph.011-47623456

ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

AMTI – NMTC - 2023 Jan. - PRIMARY – FINAL

Instructions:

1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before answering, fill in the FACE SLIP completely.
5. Your 'rough work' should be in the answer sheet itself.
6. The maximum time allowed is THREE hours.

1. Three skilled workers Akbar, Baskar and Charles are employed by a person to do three different jobs. After completion of the work the total fee the person gave to the three workers is $Rs\,6000$. It is found that $Rs.\ 400$ more than $\frac{2}{5}$ of Akbar's share, $Rs.\,200$ more than $\frac{2}{7}$ of Baskar's share and $Rs\,100$ more than $\frac{9}{17}$ of Charles' share all equal. How much did each get?

Answer (b) Akbar's Share = Rs. 1500, Baskar's Share = Rs. 2800 , Charle's share = Rs. 1700

Sol. Let us Assume that Akbar's share be \`A', Baskar's share be 'B' and charle's share be 'C'.

According to the question,

$\frac{2}{5}A + 400 = \frac{2}{7}B + 200 = \frac{9}{17}C + 100 = K \quad \ldots(1)$

As per the question, Total fee to the three persons = $Rs\,6000$

So, $A + B + C = 6000 \quad \ldots(2)$

From equation (1), we can write, $\frac{2}{5}A + 400 = k \Rightarrow A = \frac{5}{2}(k - 400) \quad \ldots(3)$

$$\frac{2}{7}B + 200 = k \Rightarrow B = \frac{7}{2}(k - 200) \quad \ldots(4)$$

$$\frac{9}{17}C + 100 = k \Rightarrow C = \frac{17}{9}(k - 100) \quad \ldots(5)$$

Substitute (3),(4) & (5) eq in eq (2)

$$\frac{5}{2}(k - 400) + \frac{7}{2}(k - 200) + \frac{17}{9}(k - 100) = 6000$$

$\Rightarrow \frac{45(k-400)+63(k-200)+34(k-100)}{18} = 6000$

$\Rightarrow 45k - 18000 + 63k - 12600 + 34k - 3400 = 6000 \times 18$

$\Rightarrow 142k - 34000 = 108000$

$\Rightarrow 142k = 142000$

$\Rightarrow K = \frac{142000}{142} = 1000$

$\Rightarrow K = 1000$

Akbar's Share = $\frac{5}{2}(k - 400) = \frac{5}{2}(1000 - 400) = \frac{5}{2} \times 600 = Rs.\,1500$

Baskar's Share = $\frac{7}{2}(k - 200) = \frac{7}{2}(1000 - 200) = \frac{7}{2} \times 800 = Rs.\,2800$

Charle's Share = $\frac{17}{9}(k - 100) = \frac{17}{9}(1000 - 100) = \frac{17}{9} \times 900 = Rs.\,1700$

2. A teacher of a primary class asked his students to calculate

$$2\frac{3}{7} \text{ of } \frac{\left(13\frac{1}{2}-9\frac{2}{3}\right)}{\left(15\frac{1}{5}-11\frac{7}{30}\right)} = A$$

The teacher has $49A$ chocolates with him. He distributed equal number of chocolates (more than one chocolate) to each student of his class irrespective of whether the students got the correct answer or not. After the distribution the teacher is left with only one chocolate and he took it. Find the maximum strength of the class.

Answer 57

Sol. Given, $A = 2\frac{3}{7}$ of $\frac{\left(13\frac{1}{2}-9\frac{2}{3}\right)}{\left(15\frac{1}{5}-11\frac{7}{30}\right)}$

$\Rightarrow A = \frac{17}{7} \times \frac{\left(\frac{27}{2}-\frac{29}{3}\right)}{\left(\frac{76}{5}-\frac{337}{30}\right)}$

Taking LCM of 2 and 3 as 6 and 30 and 5 as 30, we get

$\Rightarrow A = \frac{17}{7} \times \frac{\left(\frac{81-58}{6}\right)}{\left(\frac{456-337}{30}\right)}$

$\Rightarrow A = \frac{17}{7} \times \frac{\frac{23}{6}}{\frac{119}{30}} = \frac{17}{7} \times \frac{23}{6} \times \frac{30}{119}$

$\Rightarrow A = \frac{17}{7} \times \frac{690}{714} = \frac{115}{49}$

According to the question,

Number of chocolates $= 49A = 49 \times \frac{115}{49} = 115$

Number of chocolates distributed $= 115 - 1 = 114$ …(1)

It is given that, more than one chocolate should be distributed to each student

Thus, in that case teacher should distribute at least two chocolates

Thus, maximum strength $= \frac{No.of\ chocolates\ distributed}{chocolates\ distributed\ to\ 1\ student}$

$\Rightarrow$ Maximum strength $= \frac{114}{2} = 57$

3. In a forest, Foxes always tell the truth and jackals always lie. When seen in poor light, they are indistinguishable. A person meets three of them A, B and C, in such a poor light. He asks A, “Are you a jackal?” Although A answers his question, he could not hear it clearly. B tells him that A denied being a Jackal and C says that A really is a jackal. Among the three, how many are jackals?

Answer 1 Jackal

Sol. Given, Foxes always tell the truth.

Jackals always lie.

Two cases are possible

Case I: 'A' is a fox.

$\Rightarrow$ 'A' will deny being jackal (truth)

$\Rightarrow$ \`B' says 'A' denied being jackal, which is the truth.

So, ‘B’ is a fox.

$\Rightarrow$ 'C' said 'A' really is jackal, which is a lie.

So, \`C' is a Jackal.

Thus, their are 1 Jackal and 2 foxes.

Case II: 'A' is Jackal.

$\Rightarrow$ 'A' will say that he is not a jackal. (lie)

$\Rightarrow$ \`B' says 'A' denied being jackal, which is the truth.

So, `B' is a fox.
⇒ 'C' said 'A' really is jackal, which is the truth.
So, `C' is a fox.
Thus, agin their are 1 Jackal and 2 foxes.

4. Consider a natural number n. If n is less than 10 times the product of the digits, then n is called a $dwarf$ number. Find the number of dwarf numbers between 10 and 200.

Answer 120

Sol. For two digit number $(10 - 99)$

Let us solve for 10; $n = 10$

Product of digits $= 10(1 \times 0) = 0$

$n > 0$ (10 is not a $dwarf$ number)

Similarly, 11 is not a $dwarf$ number.

Now, $n = 12$

Product of digits $= 10(1 \times 2) = 20$

$n < 20$ (12 is a $dwarf$ number)

Let us draw a series for $dwarf$ number

$$\text{9 rows}\left\{\begin{array}{l} 12,13,14 \text{......}19 \text{ (8 numbers)} \\ 22,23,24 \text{......}29 \text{ (8 numbers)} \\ 32,33,34 \text{......}39 \text{ (8 numbers)} \\ 42,43,44 \text{......}49 \text{ (8 numbers)} \\ \vdots \\ 92,93,94 \text{......}99 \text{ (8 numbers)} \end{array}\right.$$

Total 2 − digit $dwarf$ numbers $= 9 \times 8 = 72$

For 3 − digit numbers $(100 - 200)$

From 100 to 109, no number is $dwarf$ number

As every number's digit's product is 0 and 10 times is also 0. Thus $0 < (100 ... 109)$

Similarly, $110, 111, ..., 119$ are also not $dwarf$ number

From 120 to 129 ⇒ $127, 128, 129$ (3 numbers)

From 130 to 139 ⇒ $135, 136, ..., 139$ (5 numbers)

From 140 to 149 ⇒ $144, 145, ..., 149$ (6 numbers)

From 150 to 159 ⇒ $154, 155,,,,, 159$ (6 numbers)

From 160 to 169 ⇒ $163, 164, ... 169$ (7 numbers)

From 170 to 179 ⇒ $173, 174, ..., 179$ (7 numbers)

From 180 to 189 ⇒ $183, 184, ... 189$ (7 numbers)

From 190 to 199 ⇒ $173, 174, ... 179$ (7 numbers)

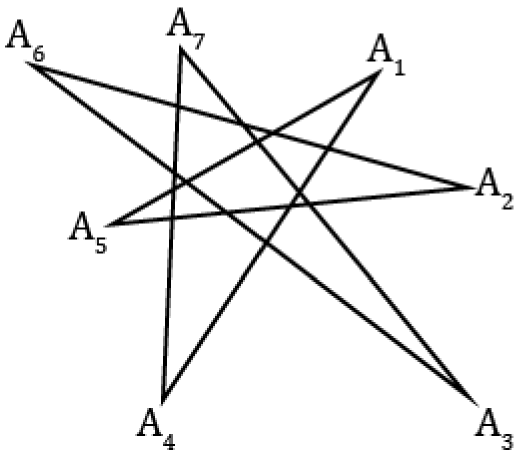
Thus, 3 − digit $dwarf$ no. $= 3 + 5 + 6 + 6 + 7 + 7 + 7 + 7$

$= 48$

Total number of $dwarf$ numbers from 10 to $200 = 72 + 48 = 120$.

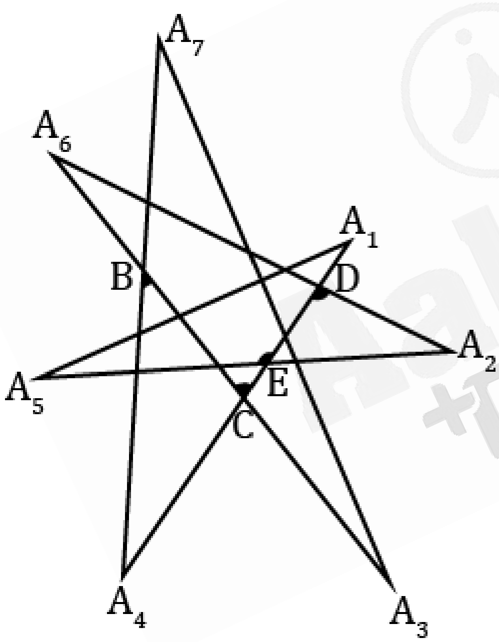
5. In the given figure, $A_1A_2A_3A_4A_5A_6A_7$ is a 7 − pointed star. Find the value of

$\angle A_1 + \angle A_2 + \angle A_3 + \angle A_4 + \angle A_5 + \angle A_6 + \angle A_7$



Answer 180^0

Sol.



We know that an exterior angle of a triangle is equal to the sum of the two opposite interior angles.

$\because \angle B$ is ext. angle of ΔBA_7A_3

$\therefore \angle B = \angle A_7 + \angle A_3$

$\because \angle C$ is ext. angle of ΔCA_4B

$\therefore \angle C = \angle A_4 + \angle B = \angle A_4 + \angle A_3 + \angle A_7$

$\because \angle D$ is ext. angle of ΔA_6CD

$\therefore \angle D = \angle C + \angle A_6 = \angle A_3 + \angle A_4 + \angle A_7 + \angle A_6$

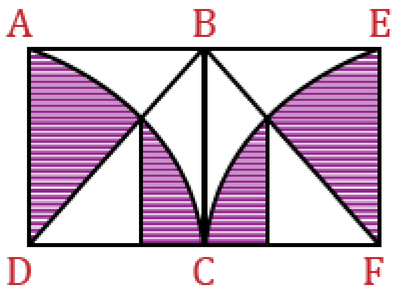
$\because \angle E$ is ext. angle of ΔDEA_2

$\therefore \angle E = \angle A_2 + D = \angle A_2 + \angle A_3 + \angle A_4 + \angle A_6 + \angle A_7$

Now, in $\Delta A_1A_5E, \angle A_1 + \angle A_5 + \angle E = 180^0$

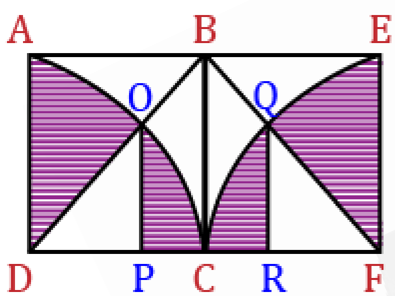
So, $\angle A_1 + \angle A_2 + \angle A_3 + \angle A_4 + \angle A_5 + \angle A_6 + \angle A_7 = 180^0$

6. Two squares of side length $20\ cm$ are joined together as in the diagram. With D, F as centers, quadrants are drawn. Taking $\pi = 3.14$, find the area of the shaded portion. Let A be the area in cm^2 Find A.



Answer $428\ cm^2$

Sol.



$\Rightarrow$ In given diagram,

$AD = OD = 20\ cm$ (Radius)

In ΔDPO or ΔFRQ, $\angle DOP = \angle ODP = 45^0$

$Sin\ 45^0 = \frac{OP}{20}$

$\frac{1}{\sqrt{2}} = \frac{OP}{20}$

$OP = \frac{20}{\sqrt{2}}$

$OP = 10\sqrt{2}\ cm$

So, $OP = QR = 10\sqrt{2}\ cm$

and $DP = RF = 10\sqrt{2}\ cm$.

Area of of shaded Region $=$ $2 \times$ Area of Quadrants $-$ $2 \times$ Area of Blank triangle

$= 2 \times \frac{1}{4}\pi \times (20)^2 - 2 \times \frac{1}{2} \times 10\sqrt{2} \times 10\sqrt{2}$

$= \frac{1}{2} \times 3.14 \times 400 - 200$

$= 3.14 \times 200 - 200$

$= 200(3.14 - 1)$

$= 200 \times 2.14$

$= 428\ cm^2$

ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC – 2023 November – JUNIOR – FINAL

Instructions:
1. Answer all questions. Each question carries 10 marks.
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3. Diagrams and justification should be given wherever necessary.
4. Before starting to answer, fill in the **FACE SLIP** completely.
5. Your 'rough work' should be done in the answer sheet itself.
6. Maximum time allowed is **THREE hours**.

Question 1:
If $b(a^2 - bc)(1 - ac) = a(b^2 - ca)(1 - bc)$ where $a \neq b$ and $abc \neq 0$, prove that
$$a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Solution:
Let's start with Left Hand Side (LHS)

L.H.S.= $b(a^2 - bc)(1 - ac) = a(b^2 - ca)(1 - bc)$

$\Rightarrow\ b(a^2 - a^3c - bc + abc^2) = a(b^2 - b^3c - ca + abc^2)$

[Multiplying the terms inside the brackets on both side]

$\Rightarrow\ a^2b - a^3bc - b^2c + ab^2c^2 = ab^2 - ab^3c - a^2c + a^2bc^2$

[Removing brackets on both side]

$\Rightarrow\ a^2b - a^3bc - b^2c + ab^2c^2 - ab^2 + ab^3c + a^2c - a^2bc^2 = 0$

[Taking all terms on one side]

$\Rightarrow\ (a^2b - ab^2) + (ab^3c - a^3bc) + (a^2c - b^2c) + (ab^2c^2 - a^2bc^2) = 0$

[Grouping the similar looking terms]

$\Rightarrow\ ab(a - b) - abc(a^2 - b^2) + c(a^2 - b^2) - abc^2(a - b) = 0$ [Taking common]

[Taking $(a - b)$ common from each term]

$\Rightarrow\ [ab - abc(a + b) + c(a + b) - abc^2] = 0$ [As $a \neq b$ & hence $a - b \neq 0$]

$\Rightarrow\ \frac{ab}{abc} - \frac{abc(a+b)}{abc} + \frac{c(a+b)}{abc} - \frac{abc^2}{abc} = \frac{0}{abc}$ [As $abc \neq 0$]

$$\Rightarrow\ \frac{1}{c} - (a + b) + \frac{(a + b)}{ab} - c = 0$$

$$\Rightarrow\ \frac{1}{c} - a - b + \frac{a}{ab} + \frac{b}{ab} - c = 0$$

$$\Rightarrow\ \frac{1}{c} - a - b + \frac{1}{b} + \frac{1}{a} - c = 0$$

$$\Rightarrow\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c$$

$\Rightarrow\ a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ = R.H.S. [Hence Proved]

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AMTI – NMTC – 2023 November – JUNIOR – FINAL

Question 2:

a, b, c are three distinct positive integers. Show that among the numbers $a^5b - ab^5$, $b^5c - bc^5$, $c^5a - ca^5$ there must be one which is divisible by 8

Solution:

Simplifying given expressions,

$a^5b - ab^5 = ab(a^4 - b^4) = ab(a^2 - b^2)(a^2 + b^2)$
$a^5b - ab^5 = ab(a + b)(a - b)(a^2 + b^2)$ …(1)

Similarly, $b^5c - bc^5 = bc(b^4 - c^4) = bc(b^2 - c^2)(b^2 + c^2)$
$b^5c - bc^5 = bc(b + c)(b - c)(b^2 + c^2)$ …(2)

And, $c^5a - ca^5 = ca(c^4 - a^4) = ca(c^2 - a^2)(c^2 + a^2)$
$c^5a - ca^5 = ca(c + a)(c - a)(c^2 + a^2)$ …(3)

Now let's assume that three distinct positive integers as
$a = 1$, $b = 3$ and $c = 5$ then
From (1),
$a^5b - ab^5 = ab(a + b)(a - b)(a^2 + b^2) = 3 \times 4 \times -2 \times 10$
$= -(3 \times 8 \times 10)$ which is divisible by 8

From (2),
$b^5c - bc^5 = bc(b + c)(b - c)(b^2 + c^2) = 15 \times 8 \times -3 \times 34$ which is divisible by 8

From (3),
$c^5a - ca^5 = ca(c + a)(c - a)(c^2 + a^2) = 5 \times 6 \times 4 \times 26$
$= 5 \times 3 \times 8 \times 26$ which is divisible by 8

Hence, among the numbers $a^5b - ab^5$, $b^5c - bc^5$, $c^5a - ca^5$ there must be one which is divisible by 8

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Question 3:

There are four points P, Q, R, S on a plane such that no three of them are collinear. Can the triangles PQR, PQS, PRS and QRS be such that at least one has an interior angle less than or equal to 45°? If so, how? If not, why?

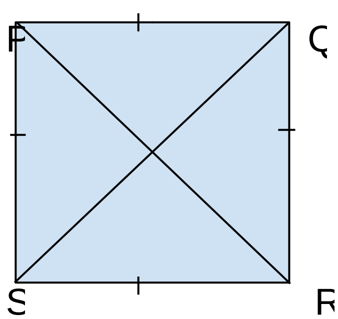
Solution.

Yes, it is possible.
Consider square PQRS and draw both diagonals. It has no three points collinear.

Consider, triangle PQR,
$\angle PQR = 90°$
$PQ = RQ$
$\Rightarrow \angle QPR = \angle QRP$ [By isosceles triangle property]

Thus, by angle sum property
$\angle PQR + \angle QPR + \angle QRP = 180°$
$\Rightarrow \angle QPR = \angle QRP = 45°$

Similarly with other triangles PQS, PRS and QRS.



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AMTI – NMTC – 2023 November – JUNIOR – FINAL

Question 4:

A straight-line l is drawn through the vertex C of an equilateral triangle ABC, wholly lying outside the triangle. AL, BM are drawn perpendiculars to the straight line l. If N is the midpoint of AB, prove that Δ LMN is an equilateral triangle.

Solution:

Triangle ABC is equilateral triangle
So,
AB = BC = AC = ‘a’ units and N is the midpoint of AB
AN = BN = $\frac{a}{2}$
CN will be the height of the equilateral triangle
CN = AL = BL = $\sqrt{\frac{3}{2}}a$
Triangle ALN is right angled triangle, by Pythagoras theorem
$LN^2 = AL^2 + AN^2$
$LN^2 = (\sqrt{\frac{3}{2}}a)^2 + (\frac{a}{2})^2$
LN = a
as AL is perpendicular to BM
AB = LM = a
LM = LN = MN
So, Δ LMN is an equilateral triangle.

Question 5:

ABCD is parallelogram. Through C, a straight line is drawn outside the parallelogram. AP, BQ and DR are drawn perpendiculars to the straight line. Show that
AP = BQ + DR. If the line through C cuts one side internally, then will the same result hold? If so, prove it. If not, what is the corresponding result? Justify the answer.

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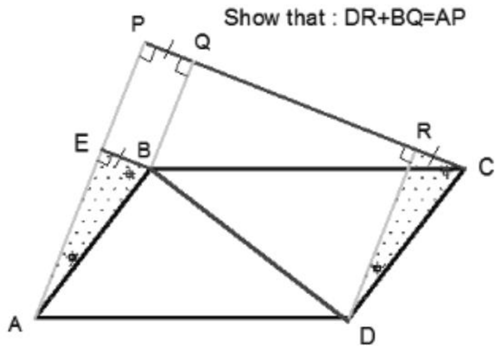
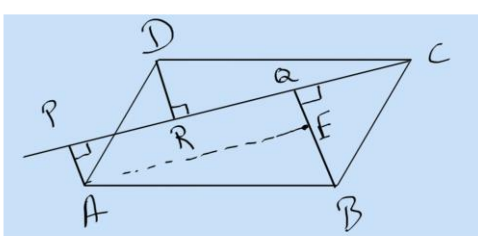
Solution:



Construction: Draw BE || CP,
In parallelogram ABCD,
AB || CD,
∠ APQ = ∠ DRC = 90 and PC is transversal.
Hence, these are co-interior angles.
⇒ AP || DR
⇒ AE || DR
⇒ ∠ ABE = ∠ DCR
⇒ ∠ BAE = ∠ CDR
Also, AB = DC,
⇒ △AEB and △DRC are congruent by ASA congruence rule.
⇒ AE = DR (by CPCT) ………… Eq(1)
Since, BE || QP, BQ || EP and BQ & EP are perpendiculars to QP then, BQPE is a rectangle,
⇒ EP = BQ ………… Eq(2)
Adding Eq(1) and Eq(2) we get: AP = DR + BQ
Now, let’s consider that line from C cuts one side internally i.e. AD.
Construction: Draw AE || PC,
Clearly from the diagram, AP < BQ.
Hence, AP ≠ BQ + DR.



ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC – 2023 November – JUNIOR – FINAL

Question 6:

m, n are non - negative real numbers whose sum is 1. Prove that the maximum and minimum values of $\frac{m^3+n^3}{m^2+n^2}$ are respectively 1 and $\frac{1}{2}$.

Solution:

Let's consider the expression $\frac{m^3+n^3}{m^2+n^2}$

Given that m and n are non-negative real numbers whose sum is 1, let's express n in terms of m using the fact that the sum of m and n is 1.

For maximum value:

Case 1: m, n = 0, 1

Substitute in the given equation

$\frac{m^3+n^3}{m^2+n^2} = \frac{0^3+1^3}{0^2+1^2} = \frac{0+1}{0+1} = \frac{1}{1} = 1$

Case 2: m, n = 1,0

Substitute in the given equation

$\frac{m^3+n^3}{m^2+n^2} = \frac{1^3+0^3}{1^2+0^2} = \frac{1+0}{1+0} = \frac{1}{1} = 1$

Case 3: m, n < 1

Statement : Cube of a number which is less than 1 is always smaller than the square of the same number.

$m^3 < m^2$ ………… Eq(1)

$n^3 < n^2$ ………… Eq(2)

Adding Eq(1) and Eq(1)

$m^3 + n^3 < m^2 + n^2$

Divide by $m^2 + n^2$ on both sides

$\frac{m^3+n^3}{m^2+n^2} < \frac{m^2+n^2}{m^2+n^2}$

$\frac{m^3+n^3}{m^2+n^2} < 1$

From case 1, 2 and 3 we can say that the maximum value of $\frac{m^3+n^3}{m^2+n^2}$ is 1.

For minimum value let us assume m = n

m + n = 1

n + n = 1

2n = 1

n $= \frac{1}{2}$

If m = $\frac{1}{2}$ then n is also equal to $\frac{1}{2}$.

Substitute the values in the given equation

$\frac{m^3+n^3}{m^2+n^2} = \frac{\frac{1}{2}^3+\frac{1}{2}^3}{\frac{1}{2}^2+\frac{1}{2}^2} = \frac{2\,x\,\frac{1}{2}^3}{2\,x\,\frac{1}{2}^2} = \left(\frac{1}{2}\right)^{3-2} = \frac{1}{2}$

So, Minimum value of $\frac{m^3+n^3}{m^2+n^2}$ is $\frac{1}{2}$.

Question 7:

(a) Solve for x: $\frac{x+5}{2018} + \frac{x+4}{2019} + \frac{x+3}{2020} + \frac{x+2}{2021} + \frac{x+1}{2022} + \frac{x}{2023}$ = - 6

(b) If $\frac{a^2+b^2}{725} = \frac{b^2+c^2}{149} = \frac{c^2+a^2}{674}$ and a - c = 18; Find the value of (a + b + c)

Solution:

a) Add 6 on both sides of the equation

$\frac{x+5}{2018} + 1 + \frac{x+4}{2019} + 1 + \frac{x+3}{2020} + 1 + \frac{x+2}{2021} + 1 + \frac{x+1}{2022} + 1 + \frac{x}{2023} + 1 = -6+6$

$\left(\frac{x+5}{2018} + \frac{2018}{2018}\right) + \left(\frac{x+4}{2019} + \frac{2019}{2019}\right) + \left(\frac{x+3}{2020} + \frac{2020}{2020}\right) + \left(\frac{x+2}{2021} + \frac{2021}{2021}\right) + \left(\frac{x+1}{2022} + \frac{2022}{2022}\right) + \left(\frac{x}{2023} + \frac{2023}{2023}\right)$

= 0

$\left(\frac{x+5+2018}{2018}\right) + \left(\frac{x+4+2019}{2019}\right) + \left(\frac{x+3+2020}{2020}\right) + \left(\frac{x+2+2021}{2021}\right) + \left(\frac{x+1+2022}{2022}\right) + \left(\frac{x+2023}{2023}\right) = 0$

$\left(\frac{x+2023}{2018}\right) + \left(\frac{x+2023}{2019}\right) + \left(\frac{x+2023}{2020}\right) + \left(\frac{x+2023}{2021}\right) + \left(\frac{x+2023}{2022}\right) + \left(\frac{x+2023}{2023}\right) = 0$

(x + 2023) $\left(\frac{1}{2018}\right) + \left(\frac{1}{2019}\right) + \left(\frac{1}{2020}\right) + \left(\frac{1}{2021}\right) + \left(\frac{1}{2022}\right) + \left(\frac{1}{2023}\right) = 0$

So, (x + 2023) = 0 ………… Eq(1)

$\left(\frac{1}{2018}\right) + \left(\frac{1}{2019}\right) + \left(\frac{1}{2020}\right) + \left(\frac{1}{2021}\right) + \left(\frac{1}{2022}\right) + \left(\frac{1}{2023}\right) = 0$ ………… Eq(2)

But the equation 2 is not possible, so we can consider

x + 2023 = 0

x = -2023

b) To solve this system of equations, we can follow a more straightforward approach by first using the information that a - c = 18 to eliminate c from the equations. Then, we can find the values of a, b, and c.

- $a^2 + b^2 = 725k^2$ ………… Eq(1)
- $b^2 + c^2 = 149k^2$ ………… Eq(2)
- $c^2 + a^2 = 674k^2$ ………… Eq(3)

Now by Eq(1) - Eq(3) we get,

$b^2 - c^2 = 51k^2$ ………… Eq(4)

$b^2 + c^2 = 149k^2$ ………… Eq(5)

Now by Eq(4) - Eq(5) we get,

$2b^2 = 200k^2$

$b^2 = 100k^2$

b = 10k

Similarly we can get the values of a and c in terms of k.

a = 25k

c = 7k

Given a - c = 18

25k - 7k = 18

18k = 18

k = 1

(a + b + c) = 25 + 10 + 7 = 42

Question 8 :

a + b + c + d = 0, then prove that
$a^3 + b^3 + c^3 + d^3 = 3(abc + bcd + cda + dab)$

Solution:

a + b + c + d = 0
a + b = -(c + d) ………… Eq(1)

Cube both side

$(a+b)^3 = [-(c+d)]^3$

$a^3 + b^3 + 3ab(a+b) = -[c^3 + d^3 + 3cd(c + d)]$

From Eq(1)

$a^3 + b^3 + 3ab[-(c + d)] = -c^3 - d^3 - 3cd[-(a + b)]$

$a^3 + b^3 - 3abc - 3abd = -c^3 - d^3 + 3acd + 3bcd$

$a^3 + b^3 + c^3 + d^3 = 3abc + 3abd + 3acd + 3bcd$

$a^3 + b^3 + c^3 + d^3 = 3(abc + bcd + cda + dab)$

Instructions:

1. Fill in the Response Sheet with your Name, Class and the Institution through which you appear, in the specified places.
2. Diagrams given are only Visual aids; they are not drawn to scale.
3. You may use separate sheets to do rough work.
4. Use of Electronic gadgets such as Calculator, Mobile Phone or Computer is not permitted.
5. Duration of the Test: 2 pm to 4 pm (2 hours).

Question 1

The value of $\sqrt{46.47.48.49 + 1}$ when simplified is

a) 2245 b) 2255 c) 2345 d) 2195

Solution: (b)

Given, $\sqrt{46.47.48.49 + 1}$

Let $a = 47,\ (a - 1) = 46,\ (a + 1) = 48\ and\ (a + 2) = 49$

$= \sqrt{(a - 1)a(a + 1)(a + 2) + 1}$

$= \sqrt{\left(a^2 + 2a\right)\left(a^2 - 1\right) + 1}$

$= \sqrt{\left(a^2 + a - 1\right)^2}$

$= a^2 + a - 1$

Substitute the value of a in the above expression, we get

$= 47^2 + 47 - 1$

$= 47(47 + 1) - 1$

$= 47\times48 - 1$

$= 2255$

Hence, the simplified value of $\sqrt{46.47.48.49 + 1}$ is 2255.

Question 2

Two regular polygons of same number of sides have side lengths 8 cm and 15 cm. The length of the side of another regular polygon of the same number of sides whose area is equal to the sum of the areas of the given polygons is (in cm.)

a) 17 b) 23 c) 38 d) 120

Solution: (a)

Let the length of the side of the 3rd polygon be x cm.

Let n denote the no. of sides of all polygons

Area of any polygon is given by = $\frac{a\times p}{n}$

Where $a = \frac{s}{2\ tan\left(\frac{180°}{n}\right)}$; $s = side\ length\ and\ p = s\times n$

Given, sum of areas of regular polygon with 8 cm and 15 cm = Area of 3rd polygon

$$\Rightarrow \frac{8n\times\frac{8}{2\ tan\left(\frac{180°}{n}\right)}}{2} + \frac{15n\times\frac{15}{2\ tan\left(\frac{180°}{n}\right)}}{2} = \frac{xn\times\frac{x}{2\ tan\left(\frac{180°}{n}\right)}}{2}$$

$$\Rightarrow \frac{64n}{2\ tan\left(\frac{180°}{n}\right)} + \frac{225n}{2\ tan\left(\frac{180°}{n}\right)} = \frac{x^2n}{2\ tan\left(\frac{180°}{n}\right)}$$

$\Rightarrow 289n = x^2n$

$\Rightarrow x^2 = 289$

$\Rightarrow x = 17$

Question 3

When $a = 2022, b = 2023$, the numerical value of $\left(\frac{a}{1+\frac{a}{b}} - \frac{b}{1-\frac{b}{a}} - \frac{2}{\frac{1}{a}-\frac{a}{b^2}}\right)$ is

a) 1 b) 2022×2023 c) $(2023)^2$ d) 0

Solution: (d)

Given, $a = 2022, b = 2023$

Substitute the values of a and b in $\left(\frac{a}{1+\frac{a}{b}} - \frac{b}{1-\frac{b}{a}} - \frac{2}{\frac{1}{a}-\frac{a}{b^2}}\right)$

$$= \left(\frac{2022}{1+\frac{2022}{2023}} - \frac{2023}{1-\frac{2023}{2022}} - \frac{2}{\frac{1}{2022}-\frac{2022}{(2023)^2}}\right)$$

$$= \left(\frac{2022\times2023}{2023+2022} - \frac{2023\times2022}{2022-2023} - \frac{2\times2022\times(2023)^2}{(2023)^2-(2022)^2}\right)$$

$$= \left(\frac{2022\times2023}{4045} - \frac{2023\times2022}{-1} - \frac{2\times2022\times(2023)^2}{(2023+2022)(2023-2022)}\right)$$

$$= \left(\frac{2022\times2023}{4045} + (2023\times2022) - \frac{2\times2022\times(2023)^2}{4045}\right)$$

$$= \frac{2022\times2023}{4045}(1 - 2\times2023) + 2023\times2022$$

$= \frac{2022\times2023}{4045}(-\ 4045)\ +\ (2023\times2022)$

$= \ \ -\ 2022\times2023\ +\ (2023\times2022)$

$=\ 0$

Hence, the numerical value of $\left(\frac{a}{1+\frac{a}{b}} - \frac{b}{1-\frac{b}{a}} - \frac{2}{\frac{1}{a}-\frac{a}{b^2}}\right)$ is 0.

Question 4

Two sides of a triangle are of lengths 5 cm and 10 cm. The length of the altitude to the third side is equal to the average of the other two altitudes. The length of the third side (in cm) is

a) 12 b) 8 c) $\frac{20}{3}$ d) 9

Solution: (c)

Let the height to the side of length 5 cm be h_1, the height to the side of length 10 be h_2, the area be A, and the height to the unknown side be h_3.

Because the area of a triangle is $\frac{bh}{2}$, we get that $5\left(h_1\right) =\ 2A$ and $10\left(h_2\right) =\ 2A$, so, setting them equal,

$h_2 = \frac{h_1}{2}$.

From the problem, we know that $2h_3 = h_1 + h_2$.

On Substituting values, we get

Thus, the third side length is going to be $\frac{2A}{0.75h_1} = \frac{5}{0.75} = \frac{20}{3}$

Hence, the length of the third side (in cm) is $\frac{20}{3}$.

Question 5

a, b, c, d, e, f are natural numbers in some order among 4, 5, 6, 12, 20, 24. The maximum value of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$ is

a) 1 b) $5\frac{1}{2}$ c) $10\frac{1}{2}$ d) 12

Solution: (d)

Let $a\ =\ 12,\ b\ =\ 6,\ c\ =\ 20,\ d\ =\ 5,\ e\ =\ 24\ and\ f\ =\ 4$

Then $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{12}{6} + \frac{20}{5} + \frac{24}{4} =\ 2\ +\ 4\ +\ 6\ =\ 12$

Hence, the maximum value of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$ is 12.

Question 6

Two consecutive natural numbers exist such that the square of their sum exceeds the sum of their squares by 112 ; then the difference of their squares is

a) 10 b) 12 c) 13 d) 15

Solution: (d)

Let the two natural numbers be n and $n + 1$

Square of the sum of those numbers = $(n + n + 1)^2 = (2n + 1)^2$

Sum of their squares = $n^2 + (n + 1)^2$

Therefore from the given data we get,

$(2n + 1)^2 = n^2 + (n + 1)^2 + 112$

$4n^2 + 4n + 1 = n^2 + n^2 + 2n + 1 + 112$

Rearrange the expression into a quadratic equation

$2n^2 + 2n - 112 = 0$

$2n^2 + 16n - 14n - 112 = 0$

$2n(n + 8) - 14(n + 8) = 0$

$(2n - 14)(n + 8) = 0$

From this we can say n is 7 or -8, but since n is a natural number, it cannot be -8.

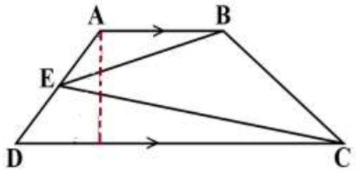
Hence, n is 7, and $n + 1 = 8$.

The numbers are 7 and 8. Now, squares of 7 and 8 are $7^2 = 49$ *and* $8^2 = 64$

Difference of their squares = $64 - 49 = 15$.

Question 7

ABCD is a trapezoid with $AB \parallel CD$. Given $AB = 11\ cm$ and $DC = 21\ cm$ and the height of the trapezoid is $4\ cm$. If E is the midpoint of AD, the area of triangle BEC (in cm^2) is



a) 32 b) 34 c) 28 d) 40

Solution: (a)

Area of trapezium = $\frac{1}{2} \times h \times (a + b) = \frac{1}{2} \times 4 \times (32) = 64\ cm^2$

We know that the area of a triangle formed by joining the midpoint of the non-parallel sides of a trapezium to the ends of the opposite sides is half of the area of a trapezium.

$\therefore Ar(\triangle BEC) \ = \frac{1}{2} \times Ar(Trap\ ABCD) \ = \frac{1}{2} \times 64 \ = 32\ cm^2$

Hence, the area of triangle BEC (in cm^2) is 32.

Question 8

One-sixth of one-fourth of three-fourths of a number is 15, the number is

a) 1020 b) 320 c) 520 d) 480

Solution: (d)

Let the number be x.
According to the question,

$$\frac{1}{6} \times \frac{1}{4} \times \frac{3}{4} \times x = 15$$

$$\Rightarrow \frac{3x}{96} = 15$$

$$\Rightarrow 3x = 15 \times 96$$

$$\Rightarrow x = \frac{1440}{3}$$

$$\Rightarrow x = 480$$

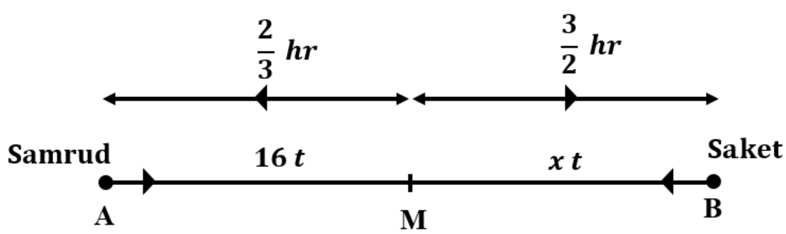
Question 9

Two places A and B are connected by a straight road. Samrud and Saket start by motorbikes respectively from A and B at the same time; after meeting each other, they complete their journey in 90 minutes and 40 minutes respectively. If the speed of Samrud's bike is 16 km/hr., then the speed of Saket's bike (in km/hr.) is …

a) 20 b) 18 c) 24 d) 22

Solution: (c)

Let Samrud and Sanket meet at point M and take *t* hrs.



Given, Time taken by Samrud from M to B = 90 mins = $\frac{3}{2}$ hr

Time taken by Saket from M to A = 40 mins = $\frac{2}{3}$ hr

Let speed of Sanket is x km/hr

As we know, Speed $= \frac{Distance}{time}$

Distance travelled by Samrud from A to M = 16 km/hr $\times$ t …(i)

Distance travelled by Sanket from B to M = x km/hr $\times$ t …(ii)

Distance travelled by Samrud from M to B = 16 km/hr $\times \frac{3}{2}$ hr = 24 km …(iii)

Distance travelled by Sanket from M to A = x km/hr $\times \frac{2}{3}$ hr …(iv)

Now, Distance travelled by Sanket from B to M = Distance travelled by Samrud from M to B

$\Rightarrow xt = 24$ (From (ii) and (iii))

$\Rightarrow t = \frac{24}{x}$ hr … (v)

Now, Distance travelled by Samrud from A to M = Distance travelled by Sanket from M to A

$\Rightarrow 16t = \frac{2}{3}x$

$\Rightarrow t = \frac{2}{48}x = \frac{1}{24}x$ …(vi)

From (v) and (vi),

$\Rightarrow \frac{24}{x} = \frac{x}{24}$

$\Rightarrow x^2 = 24^2$

$\Rightarrow x = 24$ km/hr

Question 10

The length of a rectangle is increased by 60%. By what percent should the breadth be decreased to have the same area?

a) 35.5 b) 37.5 c) 38.25 d) 36.5

Solution: (b)

Let length of rectangle =100 m

And the breadth of rectangle =100 m

As we know, area of the rectangle is (Length $\times$ Breadth)

Therefore, the original area = $100 \times 100 = 10000^2$

Given that, the length of the rectangle is increased by 60%.

First we find 60% of the length then add it to the original length to find out the new length

i.e. 100 + 60% of 100 $= 100 + \frac{60}{100} \times 100 = 160m$

And we assume that the length is decreasing at the x%

First we find x% of the breadth then subtract it to the original breadth to find out the new breadth i.e. 100 + x% of 100 $= 100 - \frac{x}{100}\times 100 = (100 - x)m$

Therefore, the new area $= 160\times(100 - x)$

According to the question the area should be same i.e.

$\Rightarrow 160\times(100 - x) = 10000$

$\Rightarrow (100 - x) = \frac{10000}{160}$

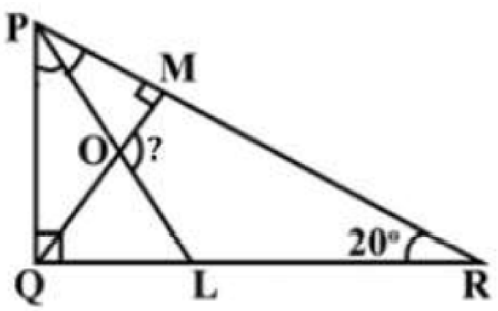
$\Rightarrow x = 100 - \frac{125}{2}$

$\therefore x = 37.5\%$

Therefore, the percent decrease in breadth is 37.5%.

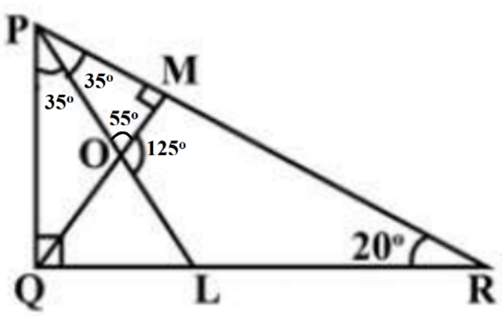
Question 11

In the adjoining figure, PL is the bisector of ∠QPR. The measure of the angle MOL is …



a) 115° b) 120° c) 125° d) 135°

Solution: (c)



Given, PL is angle bisector of $\angle QPR$.

$\angle PQR = 90^{\circ}$

from angle sum property of triangle, In ΔPQR,

$\angle PQR + \angle PRQ + \angle QPR = 180^{\circ}$

$90^{\circ} + 20^{\circ} + \angle QPR = 180^{\circ}$

$\angle QPR = 180^{\circ} - 110^{\circ}$

$\angle QPR = 70^{\circ}$

Now, $\angle QPL = \angle RPL = \frac{\angle QPR}{2} = \frac{70^{\circ}}{2} = 35^{\circ}$ (PL is angle Bisector of ∠QPR)

In $\triangle POM$,

$\angle MPO + \angle POM + \angle OMP = 180^{\circ}$

$35^{\circ} + \angle POM + 90^{\circ} = 180^{\circ}$

$\angle POM = 55^{\circ}$

Now, $\angle POM + \angle MOL = 180^{\circ}$ (linear pair)

$55^{\circ} + \angle MOL = 180^{\circ}$

$\angle MOL = 180^{\circ} - 55^{\circ}$

$\angle MOL = 125^{\circ}$

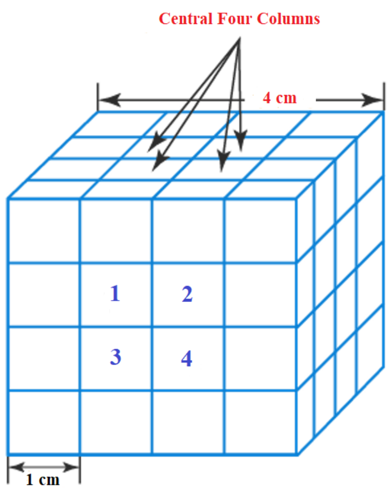
Question 12

A four centimetre cube is painted blue on all its faces. It is then cut into Identical one centimetre cubes. Among them, the number of cubes with only one face painted is …

a) 12 b) 16 c) 18 d) 24

Solution: (d)

A cube of 4 cm is shown below which is broken into sixty-four 1 cm cubes.



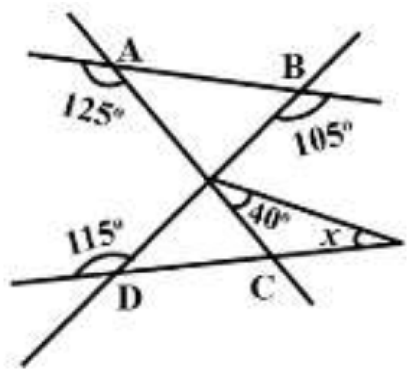
From the above diagram it is evident that the four cubes in the centre of a face of 4 cm cube do not have any of the faces painted.

The number of cubes with only one face painted
= No. of faces × Cubes painted only one face of 4 cm cube face

= 6 × 4 = 24

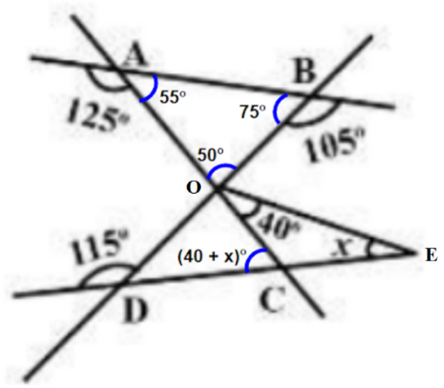
Question 13

In the adjoining figure, the value of x (in degrees) is



a) 20° b) 25° c) 30° d) 35°

Solution: (b)



$\angle OAB = 180^{\circ} - 125^{\circ} = 55^{\circ}$

in $\triangle OAB$

$\angle AOB + \angle OAB = 105^{\circ}$

$\angle AOB + 55^{\circ} = 105°$

$\angle AOB = 50^{\circ}$

Now

$\angle AOB = \angle COD = 50^{\circ}$ [Vertically opposite]

So In $\triangle COD$

$\angle COD + \angle OCD = 115^{\circ}$

$50^{\circ} + \angle OCD = 115^{\circ}$

$\angle OCD = 65^{\circ}$

Now in $\triangle OCE$
$\angle COE + \angle CEO = \angle OCD$ [External Angle]

$40^{\circ} + x = 65^{\circ}$

$x = 25^{\circ}$

Question 14

Given here is a magic square. The numerical value of $a^2 + b^2 + c^2 + d^2 + e^2$ is

a	14	b	0
c	5	6	11
4	d	10	7
15	2	e	12

a) 324 b) 144 c) 274 d) 316

Solution: (a)

In the magic square, the sum of rows and columns are the same.
$\Rightarrow 0 + 11 + 7 + 12 = 30$

$\Rightarrow 15 + 2 + e + 12 = 30$

$\Rightarrow e = 14 + d + 10 + 7 = 30$

$\Rightarrow 21 + d = 30$

$\Rightarrow d = 9c + 5 + 6 + 11 = 30$

$\Rightarrow c + 22 = 30 \Rightarrow c = 8$

Now,

$a + b + 14 = 30 \Rightarrow a + b = 16$

Now, $a + c + 4 + 15 = 30$
$\Rightarrow a + c + 19 = 30 \Rightarrow a + c = 11$

Since, $c = 8$, So,
$a + 8 = 11a = 3$

Now, $a + b = 16$
$3 + b = 16b = 13$

Now, $a^2 + b^2 + c^2 + c^2 + e^2$

$= (3)^2 + (13)^2 + (8)^2 + (9)^2 + (1)^2$

$= 9 + 169 + 64 + 81 + 1$

$= 324$

Question 15

x % of 400 added to y % of 200 gives 100. If y % of 800 is 80, what percent of x is y ?

a) 60 b) 40 c) 50 d) 20

Solution: (c)

$\frac{x}{100}\times 400 + \frac{y}{100}\times 200 = 100$

$4x + 2y = 100$............(1)

if $\frac{y}{100}\times 800 = 80$

$y = 10$

From eq. (1)

$4x + 20 = 100$

$4x = 80$

$x = 20$

$According\ to\ question$

$\frac{?}{100} \times x = y$

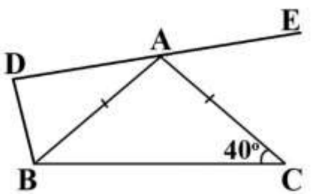
$\frac{?}{100}\times 20 = 10$

$? = 50$

FILL IN THE BLANKS:

Question 16

In the adjoining figure, $AB = AC$ and $C = 40^{\circ}$.



If $\angle ABD = (3x - 3)^{\circ}, \angle BDA = (2x + 8)^{\circ}$ and $\angle CAE = (x - 11)^{\circ}$ then $x =$_______

Solution: (21°)
$AB = AC$

So, $\angle B = \angle C = 40^{\circ}$
$In \triangle ABC$,

$\angle B + \angle C + \angle BAC = 180^{\circ}$

$40^{\circ} + 40^{\circ} + \angle BAC = 180^{\circ}$

$\angle BAC = 100^{\circ}$

In line DE, $\angle DAB + \angle BAC + \angle CAE = 180^{\circ}$,

$\angle DAB + 100^{\circ} + x - 11^{\circ} = 180^{\circ}$

$180^{\circ} - 5x - 5 + 100^{\circ} + x - 11^{\circ} = 180$

$- 4x + 84^{\circ} = 0$

$4x = 84$

$x = 21^{\circ}$

Question 17

If $a = 2022, b =- 2, c = 4044$ then the numerical value of $\frac{a(b^2-c^2)}{bc} + \frac{2b(c^2-a^2)}{ca} - \frac{c(2b^2-a^2)}{ab}$ is____

Solution: (1)
Given, $a = 2022,\ b =- 2,\ c = 4044$

$\frac{a(b^2-c^2)}{bc} + \frac{2b(c^2-a^2)}{ca} - \frac{c(2b^2-a^2)}{ab}$

$= \frac{a^2b^2-a^2c^2+2b^2c^2-2a^2b^2-2b^2c^2+a^2c^c}{abc}$

$= \frac{-a^2b^2}{abc}$

$=- \frac{ab}{c}$

$=-\frac{(2022)(-2)}{4044} = 1$

Hence, the numerical value of $\frac{a\left(b^2-c^2\right)}{bc} + \frac{2b\left(c^2-a^2\right)}{ca} - \frac{c\left(2b^2-a^2\right)}{ab}$ is 1.

Question 18

If $a = \sqrt[3]{2} - \frac{1}{\sqrt[3]{2}}$, then the numerical value of $2a^3 + 6a$ is______

Solution: (3)

$$a = (2)^{\frac{1}{3}} - \frac{1}{(2)^{\frac{1}{3}}}$$

By Cubing on both sides, we get, $a^3 = \left[(2)^{\frac{1}{3}} - \left(2^{-\frac{1}{3}}\right)\right]^3$

Use $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

$$a^3 = \left(2^{\frac{1}{3}}\right)^3 - \left(2^{-\frac{1}{3}}\right)^3 - 3\left(2^{\frac{1}{3}}\right)\left(2^{-\frac{1}{3}}\right)[2^{\frac{1}{3}} - 2^{-\frac{1}{3}})$$

$a^3 = 2 - \frac{1}{2} - 3\times a \qquad [a = (2)^{\frac{1}{3}} - \frac{1}{(2)^{\frac{1}{3}}}\ Given]$

$a^3 = 2 - \frac{1}{2} - 3a$

$a^3 + 3a = 2 - \frac{1}{2}$

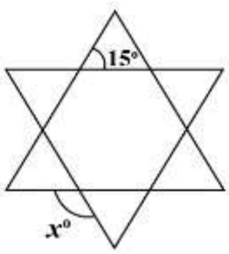
$a^3 + 3a = \frac{3}{2}$

$2a^3 + 6a = 3$

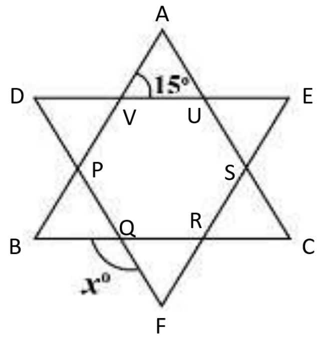
Hence, the numerical value of $2a^3 + 6a$ is 3.

Question 19

In the adjoining figure, two equilateral triangles cut each other. The measure of the angle x° is______degrees.



Solution: (165°)



Given, $\angle AVU = 15°$

$\angle AVU = DVP = 15°$ $(Vertically\ opposite\ angles)$

And also given $\Delta ABC\ is\ equilateral\ triangle\ \therefore \angle D = 60°$

Now, $\angle DPV = 180° - (60° + 15°)$

$\angle DPV = 105°$

$\therefore \angle BPQ = \angle DPV = 105^{\circ}(Vertically\ opposite\ angles)$

And $\angle B = 60^{\circ}$ (given $\triangle ABC\ is\ isosceles\ triangle$)
$\angle x$ is a exterior angle of $\triangle PBQ$

$\therefore \angle x = \angle BPQ + \angle PBQ \Rightarrow 105^{\circ} + 60^{\circ} \Rightarrow 165^{\circ}, \angle x \Rightarrow 165^{\circ}$

Hence, the measure of the angle x° is 165 degrees

Question 20

A vendor has four regular customers. He sells to the first customer half his stock of cakes and half a cake. He sells to the second customer half of the remaining stock and half a cake. He repeats this procedure for the third and the fourth customer also. Now, finally he is left with 15 cakes. The number of cakes he had in the beginning is

Solution: (289)

Let the stocks of the cake is x
$\frac{A}{\theta}$, to the first customer $\Rightarrow \frac{x}{2} + \frac{1}{2} = \frac{x+1}{2}$
to the second customer $\Rightarrow \frac{-1}{4} + \frac{1}{2} = \frac{x+1}{4}$
$to\ the\ third\ customer \Rightarrow \frac{x-3}{8} + \frac{1}{2} \Rightarrow \frac{x+1}{8}$

$to\ the\ fourth\ customer \Rightarrow \frac{x-7}{16} + \frac{1}{2} \Rightarrow \frac{x+1}{16}$

$x - \frac{x+1}{2} - \frac{x+1}{4} - \frac{x+1}{8} - \frac{x+1}{16} = 15$

$\frac{16x-8(x+1)-4(x+1)-2(x+1)-(x+1)}{16} = 15$

$\frac{16x-8x-8-4x-4-2x-2-x-1}{16} = 15$

$\frac{16x-15x-15}{16} = 15$

$x - 15 = 15\times16$

$x - 15 = 240$

$x = 240 + 15 = 255$

Hence, the number of cakes he had in the beginning is 289.

Question 21

In the sequence $1, 1, 1, 2, 1, 3, 1, 4, 1, 5$, ..., the 2022^{nd} term is______

Solution: (1011)

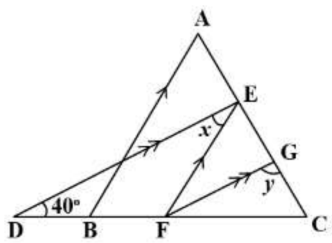
Before every natural number '1' is added in this sequence.

So, in the 10th term we are getting '5'

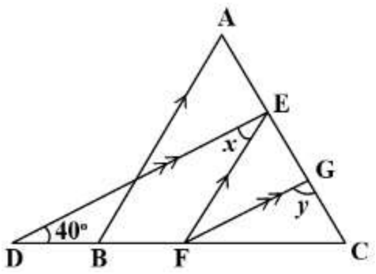
Similarly, For the 2022th term we will be getting **1011.**

Question 22

In the adjoining figure, ABC is an equilateral triangle. AB and EF are parallel. DE and FG are parallel. $\angle BDE = 40^{\circ}$. Then $x + y$ (in degrees) is_____



Solution: $(100°)$



Given, ΔABC $is\ an\ equilateral\ triangle.$

$\therefore \angle A = \angle B = \angle C = 180°$

Let, $\angle DEF = \angle EFG = x\ (alternate\ interior\ angle)$

$\angle EDF = \angle GFC = 40°\ (Corresponding\ angle)$

In, $\Delta CFG,$

$y + 40° + 60° = 180°$

$y = 80°$

According to the exterior angle property,

$y = \angle EFG + \angle GEF$

$y = x + \angle GEF$

$\angle GEF = y - x$

$60° = 80° - x$

$x = 20°$

Then $x + y = 20° + 80° = 100°$

Question 23

A gardener has to plant a number of rose plants in straight rows. First he tried 5 in each row; then he successively tried 6, 8, 9 and 12 in each row but always had 1 plant left Then he tried 13 in a row and to his pleasant surprise, no plant was left out. The smallest number of plants he could have had is

Solution: (3601)

Number of plant $= LCM(5, 6, 8, 9, 12) \times K + 1$
$N = 360K + 1$

Also it is divisible by 13

$k = 1 \rightarrow N = 361\ \ It\ is\ not\ divisible\ by\ 13$

$k = 2 \quad \rightarrow N = 721\ It\ is\ not\ divisible\ by\ 13$

$k = 3 \rightarrow N = 1081\ It\ is\ not\ divisible\ by\ 13$

$k = 10 \qquad N = 3601 \Rightarrow It\ is\ divisible\ by\ 13$

The smallest number of plants he could have had is 3601.

Question 24

A, B run a race 1 km long straight path. If A gives B 40 m start then, A wins by 19 seconds. If A gives B 30 seconds start, then B wins by 40 m. If B normally would take t_1 seconds to run the total 1 km length and A normally would take t_2 seconds to run the total 1 km length, then $t_1 - t_2$ (in seconds) is

Solution: (25)

According to the given situation,

Let a and b be the speeds of the A and B respectively,

$19 = (1000 - 40)/(b - t_2)$

$$19 = \frac{960}{1000}t_1 - t_2$$

$$or\ t_2 = \frac{960}{1000}t_1 - 19 \ldots\ldots (i)$$

$and,\ 30 = t_1 - (1000 - 40)/a$

$$30 = t_1 - \frac{960}{1000}t_2 or\ t_1 = 30 + \frac{960}{1000}t_2 \ldots\ldots\ldots (ii)$$

Subtract (i) from (ii),

$$t_1 - t_2 = -\frac{96}{100}t_1 + \frac{96}{100}t_2 + 19 + 30$$

$$t_1 - t_2 = \frac{96}{100}\left[t_2 - t_1\right] + 49$$

$$t_1 - t_2 - \frac{96}{100}\left[t_2 - t_1\right] = 49$$

$$t_1 - t_2 + \frac{96}{100}\left(t_1 - t_2\right) = 49$$

$$\left[t_1 - t_2\right]\left[1 + \frac{96}{100}\right] = 49$$

$$t_1 - t_2 = 25$$

Question 25

David computed the value of 3^{19} as $11a2261467$. He found all the digits correctly except ' a '. The value of ' a ' is

Solution: (6)

3^{19} as $11a2261467$

$1 + 1 + a + 2 + 2 + 6 + 1 + 4 + 6 + 7 = 30 + a$

Value of $a = 6$

Question 26

The sum of eight consecutive natural numbers is 124 . The sum of the next 5 natural numbers will be

Solution: (110)

Sum of eight consecutive natural numbers $= 124$

$x + x + 1 + x + 2 + x + 3 + x + 4 + x + 5 + x + 6 + x + 7 = 124$

$8x + 28 = 124$

$8x = 124 - 28$

$8x = 96$

$x = \frac{96}{8}$

$x = 12$

Sum of next five natural numbers = $x + 8 + x + 9 + x + 10 + x + 11 + x + 12$

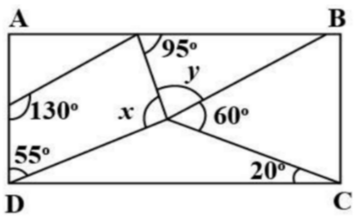
$= 12 + 8 + 12 + 9 + 12 + 10 + 12 + 11 + 12 + 12$

$= 20 + 21 + 22 + 23 + 24$

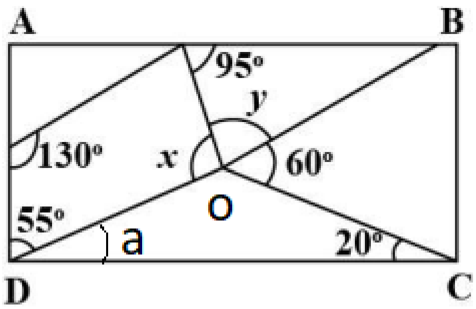
$= 110$

Question 27

In the adjoining figure, $ABCD$ is a rectangle. The value of $x + y$ (in degrees) is



Solution: (175°)



$ABCD$ is a rectangle.

$\angle D = 90^{\circ}$

$55^{\circ} + a = 90^{\circ}$

$a = 90^{\circ} - 55^{\circ}$

$a = 35^{\circ}$

In triangle DOC,.

$\angle ODC + \angle DOC + \angle OCD = 180^{\circ}$

$35^{\circ} + \angle O + 20 = 180$

$\angle O = 180 - 55 = 125^{\circ}$

We know the angle of a circle is 360° So,

$x + y + 60^{\circ} + 125^{\circ} = 360^{\circ}$

$x + y = 360^{\circ} - 185^{\circ}$

$x + y = 175^{\circ}$

Question 28

If $A = (625)^{-3/4}$ and $B = (78125)^{3/7}$, then the value of $A \times B$ is

Solution: (1)

$$A = (625)^{\frac{-3}{4}}, B = (78125)^{\frac{3}{7}}$$

$$A = \left(5^{4}\right)^{\frac{-3}{4}} = (5)^{-3} = \frac{1}{125}$$

$$B = \left(5^{7}\right)^{\frac{3}{7}} = 125$$

$A \times B \ = \frac{1}{125} \times 125 = 1$

Question 29

A room is $5\ m\ 44\ cm$ long and $3\ m\ 74\ cm$ broad. The side of the largest square-slabs which can be paved of this room (in $cm.$) is

Solution: $34\ cm$

The side of the square slab is the HCF of 544 & $374\ cm$ is 34

$544 = 2 \times 2 \times 2 \times 2 \times 2 \times 17$

$374 = 2 \times 11 \times 17$

In both Common factor is 2×17

The side of the largest square-slabs $= 34\ cm$

Question 30

A company sells umbrellas in two different sizes, large and small. This year it sold 200 umbrellas, of which one-fourth were large. The sale of large umbrellas produced one-third of the company's income. If $a : b$ is the ratio of the price of a larger umbrella to the price of a smaller umbrella, then ab^2 is

Solution: (12)

Total umbrellas = 200

Large Umbrellas = 50

Small Umbrellas = 150

Let x = total income

Income for large and small umbrellas = $\frac{x}{3} : \frac{2x}{3}$

for 1 large umbrella and 1 small umbrella $= \frac{x}{3 \times 50} : \frac{2x}{3 \times 150}$

$= \frac{x}{3} : \frac{2x}{3 \times 3}$

$= 1 : \frac{2}{3}$

$a : b = 3 : 2$

$ab^2 = 3 \times 2^2 = 12$

***Regd. Office**:* Aakash Tower, 8, Pusa Road, New Delhi-110005, Ph.011-47623456

ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC - 2023 Jan. – SUB-JUNIOR – FINAL

Instructions:

1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before answering, fill in the FACE SLIP completely.
5. Your 'rough work' should be in the answer sheet itself.
6. The maximum time allowed is THREE hours.

1. A tray contains of 40 toffees. Jaya and Uma take in turn some toffees from the tray. Each time they are allowed to take $1, 2$ or 3 toffees only. The person who gets the last toffee, wins. Jaya starts the game. Will she win? If so, how? If not, why?

Sol. Jaya and Uma both want to win.

So, Uma will try to pick the number of toffee in such a way, that Uma and Jaya get 4 toffee in every turn

$1 + 3$ or $2 + 2$ or $3 + 1$

So in 9 rounds

Uma and Jaya will pick $9 \times 4 = 36$ toffee

Now even Jaya pick 3 or 2 or 1, Uma will pick 1 or 2 or 3 toffee at last,

Hence, Uma will win.

2. Given $A = \left\{\dfrac{\frac{(a+b)^2+(a-b)^2}{b-a} - (a+b)}{\frac{1}{b-a} - \frac{1}{c-a}}\right\} \div \left\{\dfrac{(a+b)^3+(b-a)^3}{(a+b)^2+(a-b)^2}\right\}$

$B = \left\{\dfrac{c-b}{(a-b)(a-c)} - \dfrac{c-a}{(b-c)(b-a)} + \dfrac{b-a}{(c-a)(a-b)}\right\} \div \dfrac{2(a^2+b^2+c^2-ab-bc-ca)}{(a-b)(b-c)(c-a)}\}$

If $a = 2022, b = 2023$, find $(A + B)$.

Sol. $A = \left(\dfrac{\frac{2a^2+2b^2-(b^2-a^2)}{(b-a)}}{\frac{a+b-(b-a)}{(b-a)(b+a)}}\right) \div \{\dfrac{(a+b+b-a)[(a+b)^2+(b-a)^2+(a+b)(b-a)]}{2a^2+2b^2}]$

$A = \dfrac{3a^2+b^2}{(b-a)} \times \dfrac{(b-a)(b+a)}{2a} \times \dfrac{2(a^2+b^2)}{2b[3a^2+b^2]}$

$A = \dfrac{(a^2+b^2)(a+b)}{2ab}$

$B = \left\{\dfrac{b-c}{(a-b)(c-a)} + \dfrac{c-a}{(b-c)(a-b)} + \dfrac{(a-b)}{(c-a)(c-b)}\right\} \div \dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$

$B = \dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)} \times \dfrac{(a-b)(b-c)(c-a)}{(a-b)^2+(b-c)^2+(c-a)^2}$

$B = 1$

$A = \dfrac{(a^2+b^2)(a+b)}{2ab}; B = 1$

So

$A + B = \dfrac{(2022^2+2023^2)(2022+2023)}{2\times2022\times2023} + 1$

$A + B = 4046.$

3. There are square papers of areas $1, 2, 3, \ldots$ square millimeters. Asif started coloring them with red and green paint. He painted in red the papers whose areas can be written as the sum of two composite numbers and the rest in green. How many papers are painted green and what are their areas?

Sol. Areas of square papers are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\ldots\ldots$
Opposite numbers are $4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, \ldots\ldots$
Papers with green paint have the areas that can't be the sum of the two composite numbers, so areas of green painted papers are-
$1, 2, 3, 4, 5, 6, 7, 8, 9, 11$ only rest all can be the sum of two composite number lines.
$12 = 8 + 4, 13 = 4 + 9, 14 = 8 + 6, 15 = 9 + 6. 16 = 10 + 6$ and 80 on
So required sum is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 11 = 56$
Number of green painted papers $= 10$.

4. Find natural numbers a, b, c such that their sum is 6, sum of their squares is 14 and the sum of the products of a and c, and b and c is equal to the square of one more than the product of a and b.

Sol. $a + b + c = 6$ …(1)
$a^2 + b^2 + c^2 = 14$ …(2)
$ac + bc = (ab + 1)^2$ …(3)
$\Rightarrow ab + bc + ac = 11$
$\Rightarrow a^2b^2 + 1 + 2ab + ab = 11$
$\Rightarrow a^2b^2 + 3ab = 10$
$\Rightarrow a(ab^2 + 3b) = 10$

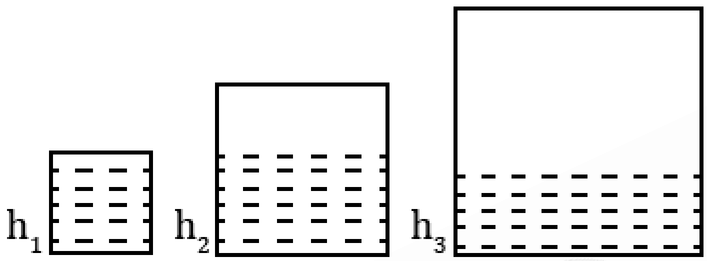
(1) $a = 5$ & $ab^2 + 3b = 2$
a, b & $c \in N$
$\Rightarrow ab^2 + 3b > 2$
$\therefore$ Rejected
OR
$a = 2$ & $ab^2 + 3b = 5$
$\Rightarrow a = 2$
$(1) \Rightarrow b + c = 4, b^2 + c^2 = 10$
Sq.$b^2 + c^2 + 2bc = 16$
$\Rightarrow bc = 3 \Rightarrow b = 3$ & $C = 1$
are $b = 1$ & $C = 3$

(2) $a = 10$ & $ab^2 + 3b = 1$
a, b & $c \in N$
$\Rightarrow ab^2 + 3b > 1$
$\therefore$ Rejected
OR
$a = 1$ & $ab^2 + 3b = 10$
$\Rightarrow a = 1$
$(1) \Rightarrow b + c = 5, b^2 + c^2 = 13$
Sq.$b^2 + c^2 + 2bc = 25$
$\Rightarrow 2bC = 12$
$\Rightarrow bc = 6$
$b = 1$ & $C = 6$ or $b = 6$ & $C = 1$
$\therefore$ Rejected as it does not satisfy equation 3
$b = 2$ & $C = 3$ it satisfies or $b = 3$ & $C = 2$ it does not satisfy

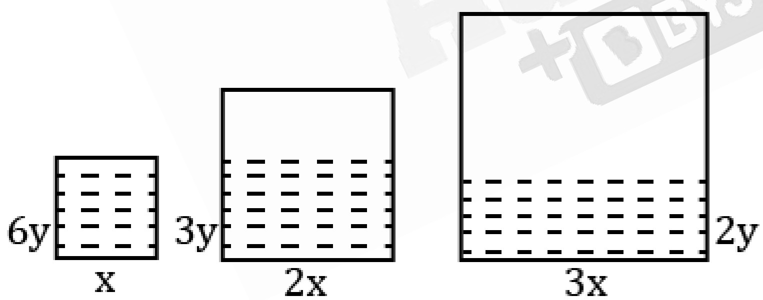
Two solutions: $a = 2, b = 1, C = 3$
$a = 1, b = 2, C = 3$

5. The volumes of three cubic containers C_1, C_2 and C_3 are in the ration $1: 8: 27$. The amount of water in them are in the ratio $1: 2: 3$. Water is poured from C_1 to C_2 and from C_2 to C_3 then the water level in all the containers is the same. Now $128\frac{4}{7}$ liters of water is poured out from C_3 to C_2 after which a certain amount is poured from C_2 to C_1 so that the depth of water in C_1 becomes twice that in C_2. This results in the amount of water in C_1 100 liters less than the original amount. How much water did each container contain originally?

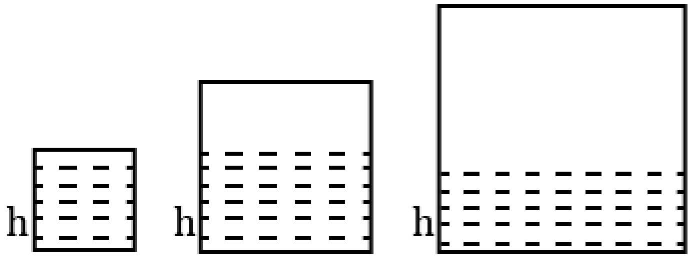
Sol. $l_1: l_2: l_3 = 1: 2: 3$



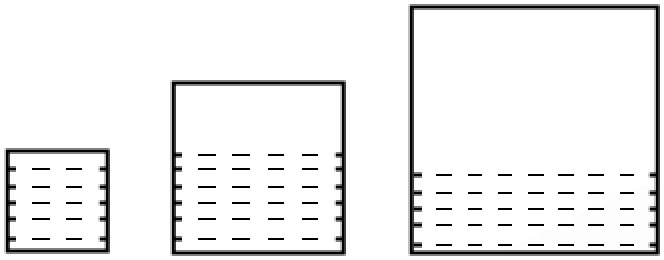
$x^2h_1: (2x)^2h_2: (3x)^2h_3 = 1: 2: 3$
$\Rightarrow h_1: 4h_2: 9h_3 = 1: 2: 3$
$\Rightarrow h_1 = 2h_2: 3h_3$
$\Rightarrow h_1: h_2: h_3 = 6: 3: 2$
Let $h_1 = 6y,\ h_2 = 3y,\ h_3 = 2y$



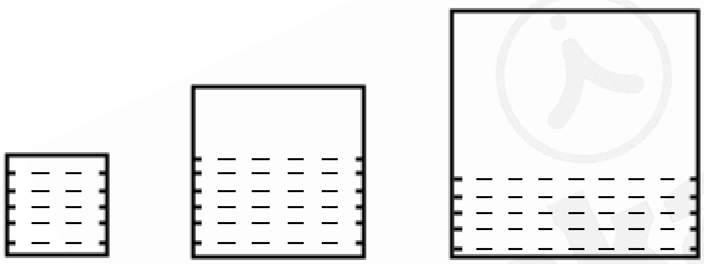
Total volume in container $= 6x^2y + 12x^2y + 18x^2y$
Now $hx^2 + h(4x^2) + h(9x^2) = 36x^2y$
$\Rightarrow 14x^2h = 36x^2y$
$\Rightarrow h = \frac{36}{14}y = \frac{18}{7}y$



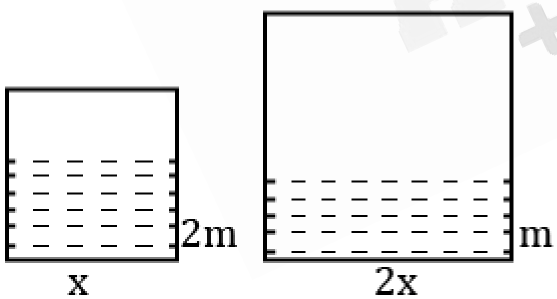
$h = \frac{18}{7}y$



$\frac{18x^2y}{7}$ $\quad$ $\frac{72x^2y}{7} + \frac{1000}{7}$ $\quad$ $\frac{162x^2y}{7} - \frac{1000}{7}$



$\frac{18}{7}x^2 \times y + z = V_1$ $\quad$ $\frac{72}{7}x^2y + \frac{1000}{7} - z = V_2$ $\quad$ $\frac{162}{7}x^2y - \frac{1000}{7}$



$V_1 = 2mx^2 \qquad V_2 = 4x^2m$

$\Rightarrow V_2 = 2V_1$

$\Rightarrow \left(\frac{72}{7}x^2y + \frac{100}{7} - z\right) = 2\left(\frac{18}{7}x^2y + z\right)$

$\Rightarrow \frac{36}{7}x^2y + \frac{100}{7} = 3z$

$\Rightarrow z = \frac{12}{7}x^2y + \frac{100}{21}$

$\Rightarrow \frac{18}{7}x^2y + \frac{12}{7}x^2y + \frac{100}{21} = 6x^2y - 100$

$\Rightarrow \frac{100}{21} + 100 = \left(6 - \frac{30}{7}\right)x^2y$

$\Rightarrow \frac{2200}{21} = \frac{12}{7}x^2y$

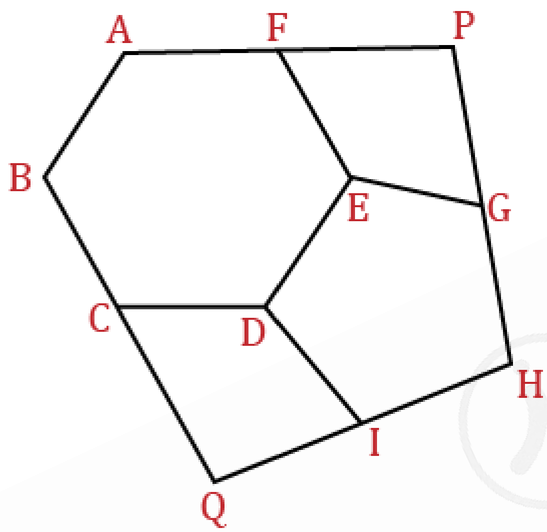
$\Rightarrow x^2y = \frac{2200\times7}{21\times12} = \frac{550}{9}$

Volume of water in C_1 is $6 \times \frac{550}{9} = \frac{1100}{3}$

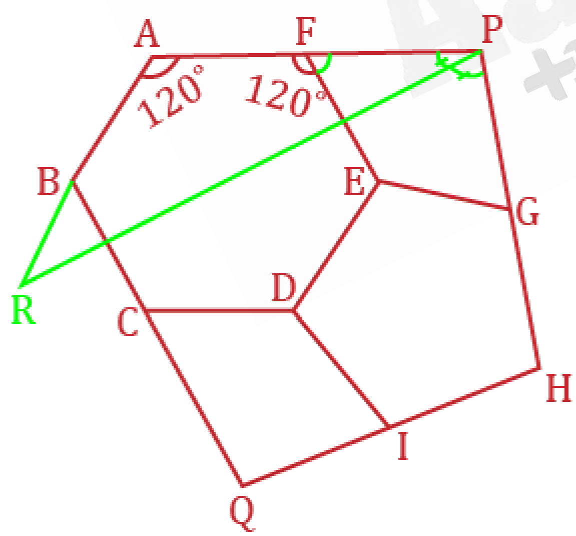
Volume of water in C_2 is $\frac{2200}{3}$

Volume of water in C_3 is $\frac{3300}{3}$.

6. In the adjoining figure, $ABCDEF$ and $DEGHI$ are respectively a regular hexagon and a regular pentagon. P is the intersection of AF and HG and similarly Q. The bisector of $\angle FPG$ cuts AB produced at R. Prove that $\angle Q = 8\angle R$.



Sol.



Each interior angle of a regular hexagon $= 120^0$

Each interior angle of a regular pentagon $= 108^0$

$\Rightarrow \angle PFE = 60^0$ & $\angle PGE = 72^0$ & $\angle FEG = 132^0$

$60^0 + \angle FPG + 72^0 + 132^0 = 360^0$

$\Rightarrow \angle FPG = 96^0$

$\Rightarrow \angle APR = 48^0$ (PR is angle bisector)

$\Rightarrow \angle R = 12^0$

By angle sum property in $APHQB$

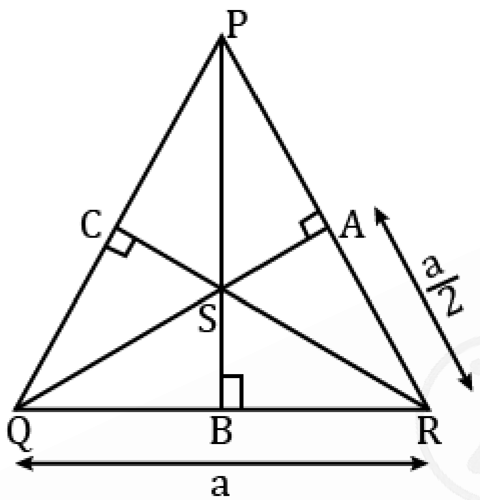
$\angle Q = 540^0 - (120^0 + 120^0 + 96 + 108^0)$

$= 96^0$

$\Rightarrow \angle Q = 8\angle R$

7. PQR is an equilateral triangle. S is any point inside the triangle. SA, SB, SC are respectively drawn perpendiculars to PR, RQ and PQ. Find the ratio of $\frac{SA+SB+SC}{QA+RB+PC}$.

Sol.



It is given that ΔPQR is an equilateral triangle. Let S be orthocentre of ΔPQR.

Let the side is a for equilateral triangle A, B, C are midpoint of sides PR, QR and PQ respectively.

$AR = \frac{a}{2} = BR = PC$

In ΔAQR, $AQ = \sqrt{a^2 - \left(\frac{a}{2}\right)^2}$

$= \sqrt{a^2 - \frac{a^2}{4}}$

$= \frac{\sqrt{3}}{2}a$

For equilateral triangle orthocentre and centroid are same point.

Therefore S divide QA in Radio $2:1$

Hence $AS = \frac{1}{3} \times \frac{\sqrt{3}}{2}a$

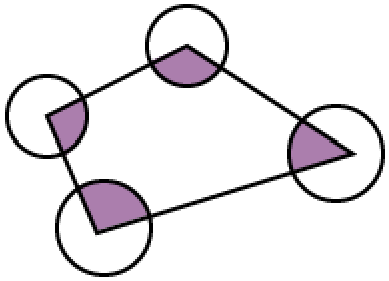
$= \frac{\sqrt{3}}{6}a$

Similarly $SB = SC = \frac{\sqrt{3}}{6}a$

Now $\frac{SA+SB+SC}{QA+RB+PC} = \frac{\frac{\sqrt{3}}{6}a+\frac{\sqrt{3}}{6}a+\frac{\sqrt{3}}{6}a}{\frac{\sqrt{3}}{2}a+\frac{a}{2}+\frac{a}{2}}$

$= \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}+1} = \frac{\sqrt{3}}{\sqrt{3}+2}$

8. In the adjoining figure, four equal circles of radius $7\,cm$ each are drawn with centers at the four vertices of a quadrilateral. Find the total area of the shaded sectors.



Sol. As the sum of all angles of a quadrilateral is 360^0,

$\therefore$ Sum of all the central angles is 360^0

Thus, total area of shaded sectors

$= \frac{360^0}{360^0} \times \pi r^2$

$= \frac{360^0}{360^0} \times \frac{22}{7} \times 7 \times 7\ cm^2$

$= 154\ cm^2$.

ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC – 2023 November – JUNIOR – FINAL

Instructions:
1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before starting to answer, fill in the **FACE SLIP** completely.
5. Your 'rough work' should be done in the answer sheet itself.
6. Maximum time allowed is **THREE hours**.

Question 1:

Find integers m, n such that the sum of their cubes is equal to the square of their sum.

Solution:

To find integers m and n such that the sum of their cubes is equal to the square of their sum, we can set up an equation and try different values until we find a solution.

The equation we want to solve is:

$m^3 + n^3 = (m + n)^2$

Now by trial-and-error method we can find different integer values for m and n. We can start with small values:

a. m = 1, n = 1

LHS = (1^3) + (1^3) = 1+1 = 2

RHS = (1 + 1)^2 = 2^2 = 4

LHS ≠ RHS

The equation is not satisfied.

b. Let's try, m = 0, n = 1:

LHS = (0^3) + (1^3) = 0+1 = 1

RHS = (0 + 1)^2 = 1^2 = 1

LHS = RHS

The equation is satisfied.

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c. Let's try, m = 1, n = 2:

LHS = (1^3) + (2^3) = 1+8 = 9

RHS = (1 + 2)^2 = 3^2 = 9

LHS = RHS

The equation is satisfied.

d. Let's try, m = 2, n = 1:

LHS = (2^3) + (1^3) = 8+1 = 9

RHS = (2+1)^2 = 3^2 = 9

LHS = RHS

The equation is satisfied.

Equation would be satisfied for the integer values, m=2 and n=2 as well.

In fact, the equation will satisfy for all the additive inverses as well, for example,

e. Let's try, m = -2, n = 2:

LHS = (-2^3) + (2^3) = -8+8 = 0

RHS = (-2+2)^2 = 0^2 = 0

LHS = RHS

The equation is satisfied.

So if we consider integers, the solution set for (m, n) will be, (0,1), (1,0), (1,2), (2,1), (2,2) and all the additive inverses such as (-1, 1), (-2, 2), etc.

Therefore, there can be infinite number of solutions.

Question 2:

In a rectangle of area 12 are placed 16 polygons, each of area 1. Show that among these polygons there are at least two which overlap in a region of area at least 1/30

Solution:

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To prove that among these 16 polygons with an area of 1 each, there are at least two that overlap in a region of at least 1/30, we can use the Pigeonhole Principle (using this principle, first we can assume something then find a contradiction against our assumption).

Let's assume that no two polygons overlap in a region of at least 1/30. We want to find a contradiction.

The area of the rectangle is 12, and each of the 16 polygons has an area of 1. If none of them overlap by at least 1/30, then the total area covered by the polygons can be at most:

Area= 16 * (1/30) = 16/30 = 8/15

This means that the area not covered by the polygons in the rectangle is at least:

12 - 8/15 = (180/15) - (8/15) = 172/15

Now, this remaining area must be covered by the polygons as well. However, this is not possible because the sum of the areas of the 16 polygons is already 16, which is greater than 172/15. This is a contradiction because the total area of the polygons cannot exceed the total area of the rectangle.

Therefore, our assumption that no two polygons overlap in a region of at least 1/30 must be false. So, there must be at least two polygons that overlap in a region of at least 1/30

Question 2:

Let a_i $(i = 1,2,3,4,5,6)$ are reals. The polynomial

$$f(x)=a_1+a_2x+a_3x^2+a_4x^3+a_5x^4+a_6x^5+7x^6-4x^7+x^8$$

can be factorized into linear factors $x-x_i$ where $i \in \{1,2,3,\ldots,8\}$.

Find the possible values of a_1.

Solution 3:

To find the possible values of a, such that the polynomial

f(x)= a1+a2x+a3x^2+a4x^3+a5x^4+a6x^5+7x^6-4x^7+x^8 can be factored into linear factors x−xi, where i belongs to the set {1, 2, 3, 4, 5, 6, 7, 8}, we can use Vieta's formulas. Specifically, we'll use the fact that the coefficient of the linear term in the polynomial (f(x) is related to the sum of the roots.

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Vieta's formulas tell us that the sum of the roots of a polynomial is equal to the negation of the coefficient of the linear term, which in this case is a2.

Therefore, we have

$$x1+x\ 2+x\ 3+x\ 4+x\ 5+x\ 6+x\ 7+x\ 8 =-a$$

Now, we also know that the sum of the roots is equal to -7 (as calculated in a previous response). Therefore, we can write:

$$-7=-a2$$

Solving for a2, we get:

$$a2=7$$

So, for the polynomial f(x) to be factorable into linear factors x−xi, where i belongs to the set 1,2,3,4,5,6,7,8, a can be any real number, and a must be equal to 7.

Question 4:

There are n (an even number) bags. Each bag contains at least one apple and at most n apples. The total number of apples is 2n. Prove that it is always possible to divide the bags into two parts such that the number of apples in each part is n.

Solution:

To prove that it's always possible to divide the bags into two parts such that the number of apples in each part is n, you can use the Pigeonhole Principle.

Start by considering the worst-case scenario, where each bag contains the maximum number of apples (n apples). In this case, you have n bags with n apples each, totalling n * n = n^2 apples.

Now, since you have 2n apples in total, you know that n^2 ≥ 2n, as stated in your problem.

Rearrange the inequality: n^2 - 2n ≥ 0.

Factor the left side of the inequality: n(n - 2) ≥ 0.

Divide both sides by n: n - 2 ≥ 0.

Add 2 to both sides: n ≥ 2.

So, you've shown that for any even number n, where n is at least 2, you have n bags with at most n apples each, and the total number of apples is 2n. By applying the Pigeonhole Principle, you've proven that it's always possible to divide the bags into two parts, each with n apples

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OR:

There is another way to solve this problem:

To start, we know that the total number of apples is 2n. This means that there must be an even number of bags that contain an even number of apples, and an even number of bags that contain an odd number of apples.

We can divide the bags into two groups: those that contain an even number of apples, and those that contain an odd number of apples. We know that the total number of apples in each group is equal to n, since the total number of apples is 2n.

Now, we can move apples between the two groups to make sure that each group has exactly n apples. To do this, we can use the following steps:

1. If there is a bag in the even group that contains more than n apples, we can move some of those apples to a bag in the odd group that contains fewer than n apples.
2. If there is a bag in the odd group that contains more than n apples, we can move some of those apples to a bag in the even group that contains fewer than n apples.
3. Continue repeating steps 1 and 2 until each bag in both groups contains exactly n apples.

We can always find a way to divide the bags into two groups such that each group has exactly n apples, because the total number of apples in each group is equal to n.

Here is an example:

Suppose we have 10 bags, and each bag contains at least one apple and at most 10 apples.
The total number of apples is 20.

We can divide the bags into two groups:

* Group 1: Bags that contain an even number of apples: {2, 4, 6, 8, 10}
* Group 2: Bags that contain an odd number of apples: {1, 3, 5, 7, 9}

We can move apples between the two groups as follows:

* Move 2 apples from bag 10 to bag 1.

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* Move 1 apple from bag 8 to bag 3.

* Move 1 apple from bag 6 to bag 5.

* Move 1 apple from bag 4 to bag 7.

* Move 1 apple from bag 2 to bag 9.

After these moves, each bag in both groups contains exactly 5 apples.

Therefore, it is always possible to divide n bags with at least one apple and at most n apples into two parts such that the number of apples in each part is n.

Question 5:

a,b,c are positive reals satisfying

$$\frac{2}{5} \leq c \leq \text{minimum}\{a,b\};\ ac \geq \frac{4}{15} \text{ and } bc \geq \frac{1}{5}.$$

Find the maximum value of (1/a +2/b+3/c).

Solution:

To find the maximum value of the expression {(1/a) + (2/b) + (3/c)}, we can use the given constraints:

2/5 <= c <= min(a, b)

ac >= 4/15

bc >= 1/5

Let's work on finding the maximum value of the expression step by step.

First, we can rewrite the expression as:

(1/a) + (2/b) + (3/c) = 1/a + 2(1/b) + 3(1/c)

Now, let's work with the constraint ac >= 4/15. We can rewrite this constraint as c >= (4/15)/a:

c >= 4/15a

Now, consider the constraint bc >= 1/5.

We can rewrite this constraint as c >= (1/5)/b:

c >= 1/5b

Since 2/5 <= c, we have a lower bound for c. We need to satisfy all these conditions while maximizing the expression.

Now, let's try to maximize the expression (1/a) + 2(1/b) + 3(1/c). To maximize this expression, we want to minimize the denominators (a, b, and c) and maximize their reciprocals (1/a, 1/b, and 1/c). So, to maximize the expression, we should have the smallest values for a, b, and c while still satisfying the given constraints.

From the constraints, we can see that c should be as small as possible since it's in the denominator. To minimize c, we can set c = 2/5 (the lower bound). This also satisfies the constraint c >= 4/15a and c >= 1/5b.

Now, let's find the values of a and b. We need to find the minimum values of a and b while satisfying the constraint 2/5 <= c <= min(a, b).

We have already set c = 2/5, so:

2/5 <= 2/5 <= min(a, b)

This implies that both a and b should be greater than or equal to 2/5.

Now, let's minimize a and b by setting a = 2/5 and b = 2/5. This ensures that all constraints are satisfied, and we've made a, b, and c as small as possible.

With these values, the expression becomes:

(1/a) + 2(1/b) + 3(1/c) = (1/(2/5)) + 2(1/(2/5)) + 3(1/(2/5)) = 5 + 10 + 15 = 30

So, the maximum value of {(1/a) + 2(1/b) + 3(1/c)} is 30, and it is achieved when a = 2/5, b = 2/5, and c = 2/5.

Question 6:

The sum of the squares of four reals x,y,z,u is 1, find the minimum value of the expression E=(x-y)(y-z)(z-u)(u-x). Find also the values of x,y,z and u when this minimum occurs

Solution:

To find the minimum value of the expression E=(x-y)(y-z)(z-u)(u-x) when the sum of the squares of the four reals x, y, z, and u is 1, we can simplify the problem using basic principles of algebra.

First, let's simplify the expression E by expanding it:

E = (x-y)(y-z)(z-u)(u-x)

Now, let's consider the sum of squares condition:

x^2 + y^2 + z^2 + u^2 = 1

We want to find the minimum value of E, so we'll first consider cases where E can be minimized:

- If all four variables are equal (x = y = z = u), E will be zero. This is the minimum value.
- If at least two of the variables are equal (e.g., x = y and z = u or x = z and y = u, etc.), E will also be zero.

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Now, let's focus on the case where none of the variables are equal. To minimize E in this case, we want to make the absolute values of each factor in E as small as possible, which means making the differences between the variables as small as possible. To do this, we can consider the following:

Let $\Delta x = |x - y|$, $\Delta y = |y - z|$, $\Delta z = |z - u|$, and $\Delta u = |u - x|$.

We want to minimize E, which is proportional to $\Delta x \Delta y \Delta z \Delta u$. To make each of these differences as small as possible, we should make the variables as close to each other as possible while still satisfying the condition $x^2 + y^2 + z^2 + u^2 = 1$.

One way to do this is to set the variables to be equally spaced along the interval [-1, 1]. For example:

$x = -1$, $y = -1/3$, $z = 1/3$, $u = 1$

This arrangement ensures that the differences between the variables are minimized. In this case, E will be the minimum value.

So, the minimum value of E is 0, and it occurs when the variables are arranged in such a way that they are equally spaced along the interval [-1, 1], as shown in the example above.

Question 7:

Let *N* be a positive integer and S(n) denote the sum of all the digits in the decimal representation of *N*. A positive integer obtained by removing one or several digits from the right hand end of the decimal representation of *N* is called a *truncation* of *N*. The sum of all truncations of *n* is denoted as T(n). Prove that S(n) + 9T(n) = *N*.

Solution 7:

Let N be a *k* digit number and N1, N2, N3, ---------, ---------, N*k* be the digits of our *k* digit number N where N*k* is at ones place, N (*k* - 1) is at tens place and so on. According to the question, their Sum is S(n).

Decimal representation of an integer implies to represent in the powers of 10 i.e.
$N = 10^{(k — 1)} \cdot N1 + 10^{(k — 2)} \cdot N2 + --- + ----- + 10 \cdot N(k — 1) + Nk$,
$\Rightarrow N = S + 9 \cdot [\{ 10^{(k — 1)} \cdot N1 — 1\} + \{ 10^{(k — 1)} \cdot N2 — 1\} + ------------+ ------ + N(k — 1)]$
$= S(n) + 9 \cdot T(n)$.
Hence, proved.

Question 8:

ABCD is a cyclic quadrilateral. the mid points of the diagonals AC and BD are respectively P and Q. If BD bisects the ∠ AQC, then prove that AC will bisect ∠ BPD.

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Solution:

Given:

ABCD is a cyclic quadrilateral.

The midpoints of diagonal AC and BD are P and Q, respectively.

BD bisects ∠ AQC.

To prove:

AC bisects ∠ BPD

Proof:

In a cyclic quadrilateral (a four-sided figure with all its vertices on a circle), we know that opposite ∠s add up to 180 degrees. This means that ∠ A + ∠ C = 180 degrees, and ∠ B + ∠ D = 180 degrees.

We also know that P is the midpoint of diagonal AC, and Q is the midpoint of diagonal BD.

Now, we are given that BD bisects ∠ AQC. This means that ∠ BQD is equal to ∠ DQC.

Let's consider the triangles in our quadrilateral:

Triangle BQC has BQ and QC as equal sides because Q is the midpoint of BD.

Triangle DQC has DQ and QC as equal sides because Q is the midpoint of BD.

Since BQ is equal to DQ, and QC is common to both triangles, by the Side-Side-Side (SSS) rule, we can say that triangles BQC is congruent (exactly the same) to triangles DQC.

Now, if two triangles are congruent, their corresponding ∠s are equal. Therefore, ∠ BQC is equal to ∠ DQC.

Now, let's look at the points P and C in our quadrilateral. Since P is the midpoint of diagonal AC, it divides AC into two equal parts: AP and PC.

In a similar way to our tri∠s above, we can see that triangles APC is congruent to triangles CPC (by Side-Side-Side), and their corresponding ∠s are equal. This means ∠ APC is equal to ∠ CPC.

Now, consider the ∠s in our quadrilateral:

∠ BPD can be split into two parts: ∠ BPC + ∠ CPC.

$\angle$ BPC + $\angle$ CPC = $\angle$ BQC + $\angle$ CPC (as we proved that $\angle$ BQC = $\angle$ DQC).
Now, we have:

$\angle$ BPD = $\angle$ BQC + $\angle$ CPC = $\angle$ DQC + $\angle$ CPC

But we know that $\angle$ DQC + $\angle$ CPC is equal to $\angle$ DQC + $\angle$ CPC in the other triangle.

So, $\angle$ BPD = $\angle$ DQC + $\angle$ CPC = $\angle$ DQC + $\angle$ CPC

This means that $\angle$ BPD can be split into two equal parts, so AC bisects $\angle$ BPD.

Hence, we have proved that AC bisects $\angle$ BPD in our given cyclic quadrilateral ABCD.

Instructions:

1. Fill in the Response Sheet with your Name, Class and the Institution through which you appear, in the specified places.
2. Diagrams given are only Visual aids; they are not drawn to scale.
3. You may use separate sheets to do rough work.
4. Use of Electronic gadgets such as Calculator, Mobile Phone or Computer is not permitted.
5. Duration of the Test: 2 pm to 4 pm (2 hours).

Question 1

The value of $\sqrt{46.47.48.49 + 1}$ when simplified is

a) 2245 b) 2255 c) 2345 d) 2195

Solution: (b)

Given, $\sqrt{46.47.48.49 + 1}$

Let $a = 47,\ (a - 1) = 46,\ (a + 1) = 48\ and\ (a + 2) = 49$

$= \sqrt{(a - 1)a(a + 1)(a + 2) + 1}$

$= \sqrt{\left(a^2 + 2a\right)\left(a^2 - 1\right) + 1}$

$= \sqrt{(a^2 + a - 1)^2}$

$= a^2 + a - 1$

Substitute the value of a in the above expression, we get

$= 47^2 + 47 - 1$

$= 47(47 + 1) - 1$

$= 47\times48 - 1$

$= 2255$

Hence, the simplified value of $\sqrt{46.47.48.49 + 1}$ is 2255.

Question 2

Two regular polygons of same number of sides have side lengths 8 cm and 15 cm. The length of the side of another regular polygon of the same number of sides whose area is equal to the sum of the areas of the given polygons is (in cm.)

a) 17 b) 23 c) 38 d) 120

Solution: (a)

Let the length of the side of the 3rd polygon be x cm.

Let n denote the no. of sides of all polygons

Area of any polygon is given by = $\frac{a\times p}{n}$

Where $a = \frac{s}{2\tan\left(\frac{180°}{n}\right)}$; $s = side\ length\ and\ p = s\times n$

Given, sum of areas of regular polygon with 8 cm and 15 cm = Area of 3rd polygon

$$\Rightarrow \frac{8n\times\frac{8}{2\tan\left(\frac{180°}{n}\right)}}{2} + \frac{15n\times\frac{15}{2\tan\left(\frac{180°}{n}\right)}}{2} = \frac{xn\times\frac{x}{2\tan\left(\frac{180°}{n}\right)}}{2}$$

$$\Rightarrow \frac{64n}{2\tan\left(\frac{180°}{n}\right)} + \frac{225n}{2\tan\left(\frac{180°}{n}\right)} = \frac{x^2 n}{2\tan\left(\frac{180°}{n}\right)}$$

$\Rightarrow 289n = x^2 n$

$\Rightarrow x^2 = 289$

$\Rightarrow x = 17$

Question 3

When $a = 2022, b = 2023$, the numerical value of $\left(\frac{a}{1+\frac{a}{b}} - \frac{b}{1-\frac{b}{a}} - \frac{2}{\frac{1}{a}-\frac{a}{b^2}}\right)$ is

a) 1 b) 2022×2023 c) $(2023)^2$ d) 0

Solution: (d)

Given, $a = 2022, b = 2023$

Substitute the values of a and b in $\left(\frac{a}{1+\frac{a}{b}} - \frac{b}{1-\frac{b}{a}} - \frac{2}{\frac{1}{a}-\frac{a}{b^2}}\right)$

$$= \left(\frac{2022}{1+\frac{2022}{2023}} - \frac{2023}{1-\frac{2023}{2022}} - \frac{2}{\frac{1}{2022}-\frac{2022}{(2023)^2}}\right)$$

$$= \left(\frac{2022\times2023}{2023+2022} - \frac{2023\times2022}{2022-2023} - \frac{2\times2022\times(2023)^2}{(2023)^2-(2022)^2}\right)$$

$$= \left(\frac{2022\times2023}{4045} - \frac{2023\times2022}{-1} - \frac{2\times2022\times(2023)^2}{(2023+2022)(2023-2022)}\right)$$

$$= \left(\frac{2022\times2023}{4045} + (2023\times2022) - \frac{2\times2022\times(2023)^2}{4045}\right)$$

$$= \frac{2022\times2023}{4045}(1 - 2\times2023) + 2023\times2022$$

$= \frac{2022\times2023}{4045}(-\ 4045)\ +\ (2023\times2022)$

$= \ -\ 2022\times2023\ +\ (2023\times2022)$

$= 0$

Hence, the numerical value of $\left(\frac{a}{1+\frac{a}{b}} - \frac{b}{1-\frac{b}{a}} - \frac{2}{\frac{1}{a}-\frac{a}{b^2}}\right)$ is 0.

Question 4

Two sides of a triangle are of lengths 5 cm and 10 cm. The length of the altitude to the third side is equal to the average of the other two altitudes. The length of the third side (in cm) is

a) 12 b) 8 c) $\frac{20}{3}$ d) 9

Solution: (c)

Let the height to the side of length 5 cm be h_1, the height to the side of length 10 be h_2, the area be A, and the height to the unknown side be h_3.

Because the area of a triangle is $\frac{bh}{2}$, we get that $5\left(h_1\right) = 2A$ and $10\left(h_2\right) = 2A$, so, setting them equal,

$h_2 = \frac{h_1}{2}$.

From the problem, we know that $2h_3 = h_1 + h_2$.

On Substituting values, we get

Thus, the third side length is going to be $\frac{2A}{0.75h_1} = \frac{5}{0.75} = \frac{20}{3}$

Hence, the length of the third side (in cm) is $\frac{20}{3}$.

Question 5

a, b, c, d, e, f are natural numbers in some order among 4, 5, 6, 12, 20, 24. The maximum value of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$ is

a) 1 b) $5\frac{1}{2}$ c) $10\frac{1}{2}$ d) 12

Solution: (d)

Let $a = 12,\ b = 6,\ c = 20,\ d = 5,\ e = 24\ and\ f = 4$

Then $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{12}{6} + \frac{20}{5} + \frac{24}{4} = 2 + 4 + 6 = 12$

Hence, the maximum value of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$ is 12.

Question 6

Two consecutive natural numbers exist such that the square of their sum exceeds the sum of their squares by 112 ; then the difference of their squares is

a) 10 b) 12 c) 13 d) 15

Solution: (d)

Let the two natural numbers be n and $n + 1$

Square of the sum of those numbers = $(n + n + 1)^2 = (2n + 1)^2$

Sum of their squares = $n^2 + (n + 1)^2$

Therefore from the given data we get,

$(2n + 1)^2 = n^2 + (n + 1)^2 + 112$

$4n^2 + 4n + 1 = n^2 + n^2 + 2n + 1 + 112$

Rearrange the expression into a quadratic equation

$2n^2 + 2n - 112 = 0$

$2n^2 + 16n - 14n - 112 = 0$

$2n(n + 8) - 14(n + 8) = 0$

$(2n - 14)(n + 8) = 0$

From this we can say n is 7 or -8, but since n is a natural number, it cannot be -8.

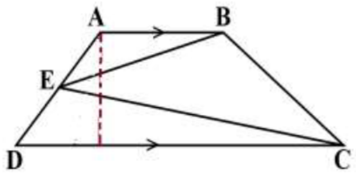
Hence, n is 7, and $n + 1 = 8$.

The numbers are 7 and 8. Now, squares of 7 and 8 are $7^2 = 49$ *and* $8^2 = 64$

Difference of their squares = $64 - 49 = 15$.

Question 7

ABCD is a trapezoid with $AB \parallel CD$. Given $AB = 11\ cm$ and $DC = 21\ cm$ and the height of the trapezoid is $4\ cm$. If E is the midpoint of AD, the area of triangle BEC (in cm^2) is



a) 32 b) 34 c) 28 d) 40

Solution: (a)

Area of trapezium $= \frac{1}{2} \times h \times (a + b) = \frac{1}{2} \times 4 \times (32) = 64\ cm^2$

We know that the area of a triangle formed by joining the midpoint of the non-parallel sides of a trapezium to the ends of the opposite sides is half of the area of a trapezium.

$\therefore Ar(\triangle BEC) \ = \frac{1}{2} \times Ar(\ Trap\ ABCD) \ = \frac{1}{2} \times 64 \ = 32\ cm^2$

Hence, the area of triangle BEC (in cm^2) is 32.

Question 8

One-sixth of one-fourth of three-fourths of a number is 15, the number is

a) 1020 b) 320 c) 520 d) 480

Solution: (d)
Let the number be x.
According to the question,

$$\frac{1}{6} \times \frac{1}{4} \times \frac{3}{4} \times x = 15$$

$$\Rightarrow \frac{3x}{96} = 15$$

$$\Rightarrow 3x = 15 \times 96$$

$$\Rightarrow x = \frac{1440}{3}$$

$$\Rightarrow x = 480$$

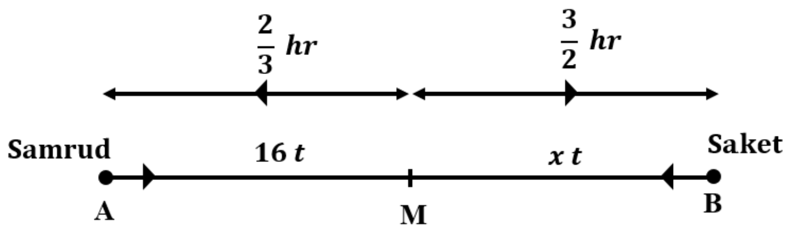
Question 9

Two places A and B are connected by a straight road. Samrud and Saket start by motorbikes respectively from A and B at the same time; after meeting each other, they complete their journey in 90 minutes and 40 minutes respectively. If the speed of Samrud's bike is 16 km/hr., then the speed of Saket's bike (in km/hr.) is …

a) 20 b) 18 c) 24 d) 22

Solution: (c)

Let Samrud and Sanket meet at point M and take *t* hrs.



Given, Time taken by Samrud from M to B = 90 mins = $\frac{3}{2}$ hr

Time taken by Saket from M to A = 40 mins = $\frac{2}{3}$ hr

Let speed of Sanket is x km/hr
As we know, Speed $= \frac{Distance}{time}$
Distance travelled by Samrud from A to M = 16 km/hr $\times$ t …(i)
Distance travelled by Sanket from B to M = x km/hr $\times$ t …(ii)
Distance travelled by Samrud from M to B = 16 km/hr $\times \frac{3}{2}$ hr = 24 km …(iii)

Distance travelled by Sanket from M to A = x km/hr $\times \frac{2}{3}$ hr …(iv)
Now, Distance travelled by Sanket from B to M = Distance travelled by Samrud from M to B

$\Rightarrow xt = 24$ (From (ii) and (iii))

$\Rightarrow t = \frac{24}{x}$ hr … (v)

Now, Distance travelled by Samrud from A to M = Distance travelled by Sanket from M to A

$\Rightarrow 16t = \frac{2}{3}x$

$\Rightarrow t = \frac{2}{48}x = \frac{1}{24}x$ …(vi)

From (v) and (vi),

$\Rightarrow \frac{24}{x} = \frac{x}{24}$

$\Rightarrow x^2 = 24^2$

$\Rightarrow x = 24$ km/hr

Question 10

The length of a rectangle is increased by 60%. By what percent should the breadth be decreased to have the same area?

a) 35.5 b) 37.5 c) 38.25 d) 36.5

Solution: (b)

Let length of rectangle =100 m
And the breadth of rectangle =100 m
As we know, area of the rectangle is (Length $\times$ Breadth)

Therefore, the original area = $100\times 100 = 10000^2$

Given that, the length of the rectangle is increased by 60%.
First we find 60% of the length then add it to the original length to find out the new length
i.e. 100 + 60% of 100 $= 100 + \frac{60}{100}\times 100 = 160m$

And we assume that the length is decreasing at the x%

First we find x% of the breadth then subtract it to the original breadth to find out the new breadth i.e. 100 + x% of 100 $= 100 - \frac{x}{100} \times 100 = (100 - x)m$

Therefore, the new area $= 160 \times (100 - x)$

According to the question the area should be same i.e.

$\Rightarrow 160 \times (100 - x) = 10000$

$\Rightarrow (100 - x) = \frac{10000}{160}$

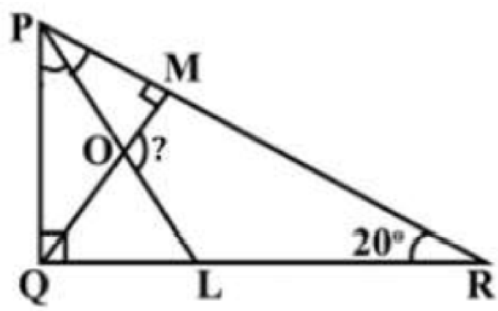
$\Rightarrow x = 100 - \frac{125}{2}$

$\therefore x = 37.5\%$

Therefore, the percent decrease in breadth is 37.5%.

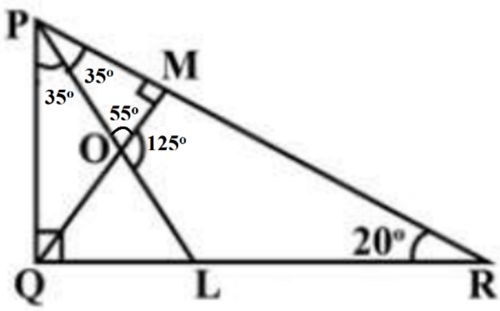
Question 11

In the adjoining figure, PL is the bisector of ∠QPR. The measure of the angle MOL is …



a) 115° b) 120° c) 125° d) 135°

Solution: (c)



Given, PL is angle bisector of $\angle QPR$.

$\angle PQR = 90^{\circ}$

from angle sum property of triangle, In ΔPQR,

$\angle PQR + \angle PRQ + \angle QPR = 180^{\circ}$

$90^{\circ} + 20^{\circ} + \angle QPR = 180^{\circ}$

$\angle QPR = 180^{\circ} - 110^{\circ}$

$\angle QPR = 70^{\circ}$

Now, $\angle QPL = \angle RPL = \frac{\angle QPR}{2} = \frac{70^{\circ}}{2} = 35^{\circ}$ (PL is angle Bisector of ∠QPR)

In $\triangle POM,$

$\angle MPO + \angle POM + \angle OMP = 180^{\circ}$

$35^{\circ} + \angle POM + 90^{\circ} = 180^{\circ}$

$\angle POM = 55^{\circ}$

Now, $\angle POM + \angle MOL = 180^{\circ}$ (linear pair)

$55^{\circ} + \angle MOL = 180^{\circ}$

$\angle MOL = 180^{\circ} - 55^{\circ}$

$\angle MOL = 125^{\circ}$

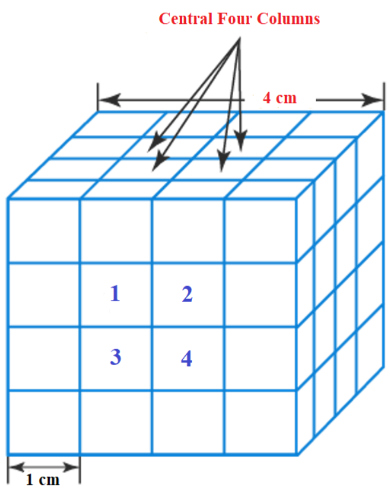
Question 12

A four centimetre cube is painted blue on all its faces. It is then cut into Identical one centimetre cubes. Among them, the number of cubes with only one face painted is …

a) 12 b) 16 c) 18 d) 24

Solution: (d)

A cube of 4 cm is shown below which is broken into sixty-four 1 cm cubes.



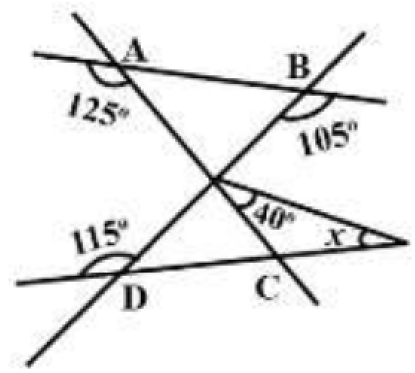
From the above diagram it is evident that the four cubes in the centre of a face of 4 cm cube do not have any of the faces painted.

The number of cubes with only one face painted
= No. of faces × Cubes painted only one face of 4 cm cube face

= 6 × 4 = 24

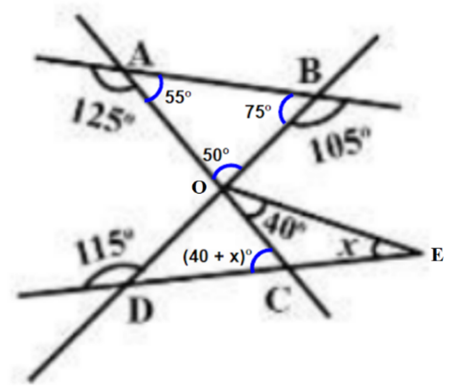
Question 13

In the adjoining figure, the value of x (in degrees) is



a) 20° b) 25° c) 30° d) 35°

Solution: (b)



$\angle OAB = 180^{\circ} - 125^{\circ} = 55^{\circ}$

in $\triangle OAB$

$\angle AOB + \angle OAB = 105^{\circ}$

$\angle AOB + 55^{\circ} = 105°$

$\angle AOB = 50^{\circ}$

Now

$\angle AOB = \angle COD = 50^{\circ}$ [Vertically opposite]

So In $\triangle COD$

$\angle COD + \angle OCD = 115^{\circ}$

$50^{\circ} + \angle OCD = 115^{\circ}$

$\angle OCD = 65^{\circ}$

Now in $\triangle OCE$
$\angle COE + \angle CEO = \angle OCD$ [External Angle]

$40^{\circ} + x = 65^{\circ}$

$x = 25^{\circ}$

Question 14

Given here is a magic square. The numerical value of $a^2 + b^2 + c^2 + d^2 + e^2$ is

a	14	**b**	0
c	5	6	11
4	***d***	10	7
15	2	***e***	12

a) 324 b) 144 c) 274 d) 316

Solution: (a)

In the magic square, the sum of rows and columns are the same.
$\Rightarrow 0 + 11 + 7 + 12 = 30$

$\Rightarrow 15 + 2 + e + 12 = 30$

$\Rightarrow e = 14 + d + 10 + 7 = 30$

$\Rightarrow 21 + d = 30$

$\Rightarrow d = 9c + 5 + 6 + 11 = 30$

$\Rightarrow c + 22 = 30 \Rightarrow c = 8$

Now,

$a + b + 14 = 30 \Rightarrow a + b = 16$

Now, $a + c + 4 + 15 = 30$
$\Rightarrow a + c + 19 = 30 \Rightarrow a + c = 11$

Since, $c = 8$, So,
$a + 8 = 11a = 3$

Now, $a + b = 16$
$3 + b = 16b = 13$

Now, $a^2 + b^2 + c^2 + c^2 + e^2$

$= (3)^2 + (13)^2 + (8)^2 + (9)^2 + (1)^2$

$= 9 + 169 + 64 + 81 + 1$

$= 324$

Question 15

x % of 400 added to y % of 200 gives 100. If y % of 800 is 80, what percent of x is y ?

a) 60 b) 40 c) 50 d) 20

Solution: (c)

$\frac{x}{100}\times 400 + \frac{y}{100}\times 200 = 100$

$4x + 2y = 100$............(1)

if $\frac{y}{100}\times 800 = 80$

$y = 10$

From eq. (1)

$4x + 20 = 100$

$4x = 80$

$x = 20$

$According\ to\ question$

$\frac{?}{100}\times x = y$

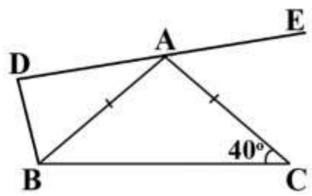
$\frac{?}{100}\times 20 = 10$

$? = 50$

FILL IN THE BLANKS:

Question 16

In the adjoining figure, $AB = AC$ and $C = 40^{\circ}$.



If $\angle ABD = (3x - 3)^{\circ}, \angle BDA = (2x + 8)^{\circ}$ and $\angle CAE = (x - 11)^{\circ}$ then $x =$_______

Solution: (21°)
$AB = AC$

So, $\angle B = \angle C = 40^{\circ}$
$In\triangle ABC$,

$\angle B + \angle C + \angle BAC \ = 180^{\circ}$

$40^{\circ} + 40^{\circ} + \angle BAC \ = 180^{\circ}$

$\angle BAC = 100^{\circ}$

In line DE, $\angle DAB + \angle BAC + \angle CAE = 180^{\circ}$,

$\angle DAB + 100^{\circ} + x - 11^{\circ} = 180^{\circ}$

$180^{\circ} - 5x - 5 + 100^{\circ} + x - 11^{\circ} = 180$

$- 4x + 84^{\circ} = 0$

$4x = 84$

$x = 21^{\circ}$

Question 17

If $a = 2022, b =- 2, c = 4044$ then the numerical value of $\frac{a(b^2-c^2)}{bc} + \frac{2b(c^2-a^2)}{ca} - \frac{c(2b^2-a^2)}{ab}$ is____

Solution: (1)
Given, $a = 2022,\ b =- 2,\ c = 4044$

$\frac{a(b^2-c^2)}{bc} + \frac{2b(c^2-a^2)}{ca} - \frac{c(2b^2-a^2)}{ab}$

$= \frac{a^2b^2-a^2c^2+2b^2c^2-2a^2b^2-2b^2c^2+a^2c^c}{abc}$

$= \frac{-a^2b^2}{abc}$

$=- \frac{ab}{c}$

$= -\frac{(2022)(-2)}{4044} = 1$

Hence, the numerical value of $\frac{a(b^2-c^2)}{bc} + \frac{2b(c^2-a^2)}{ca} - \frac{c(2b^2-a^2)}{ab}$ is 1.

Question 18

If $a = \sqrt[3]{2} - \frac{1}{\sqrt[3]{2}}$, then the numerical value of $2a^3 + 6a$ is______

Solution: (3)

$$a = (2)^{\frac{1}{3}} - \frac{1}{(2)^{\frac{1}{3}}}$$

By Cubing on both sides, we get, $a^3 = \left[(2)^{\frac{1}{3}} - \left(2^{-\frac{1}{3}}\right)\right]^3$

Use $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

$$a^3 = \left(2^{\frac{1}{3}}\right)^3 - \left(2^{-\frac{1}{3}}\right)^3 - 3\left(2^{\frac{1}{3}}\right)\left(2^{-\frac{1}{3}}\right)[2^{\frac{1}{3}} - 2^{-\frac{1}{3}})$$

$a^3 = 2 - \frac{1}{2} - 3\times a \qquad [a = (2)^{\frac{1}{3}} - \frac{1}{(2)^{\frac{1}{3}}}\ Given]$

$a^3 = 2 - \frac{1}{2} - 3a$

$a^3 + 3a = 2 - \frac{1}{2}$

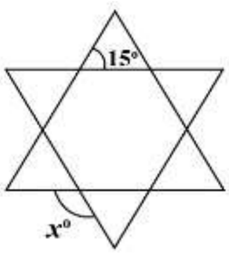
$a^3 + 3a = \frac{3}{2}$

$2a^3 + 6a = 3$

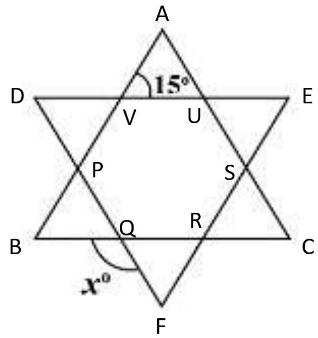
Hence, the numerical value of $2a^3 + 6a$ is 3.

Question 19

In the adjoining figure, two equilateral triangles cut each other. The measure of the angle x° is______degrees.



Solution: (165°)



Given, $\angle AVU = 15°$

$\angle AVU = DVP = 15°$ $(Vertically\ opposite\ angles)$

And also given $\Delta ABC\ is\ equilateral\ triangle\ \therefore \angle D = 60°$

Now, $\angle DPV = 180° - (60° + 15°)$

$\angle DPV = 105°$

$\therefore \angle BPQ = \angle DPV = 105^{\circ}(Vertically\ opposite\ angles)$

And $\angle B = 60^{\circ}$ (given $\triangle ABC\ is\ isosceles\ triangle$)
$\angle x$ is a exterior angle of $\triangle PBQ$

$\therefore \angle x = \angle BPQ + \angle PBQ \Rightarrow 105^{\circ} + 60^{\circ} \Rightarrow 165^{\circ}, \angle x \Rightarrow 165^{\circ}$

Hence, the measure of the angle x° is 165 degrees

Question 20

A vendor has four regular customers. He sells to the first customer half his stock of cakes and half a cake. He sells to the second customer half of the remaining stock and half a cake. He repeats this procedure for the third and the fourth customer also. Now, finally he is left with 15 cakes. The number of cakes he had in the beginning is

Solution: (289)

Let the stocks of the cake is x

$\frac{A}{\theta}$, to the first customer $\Rightarrow \frac{x}{2} + \frac{1}{2} = \frac{x+1}{2}$

to the second customer $\Rightarrow \frac{-1}{4} + \frac{1}{2} = \frac{x+1}{4}$

$to\ the\ third\ customer \Rightarrow \frac{x-3}{8} + \frac{1}{2} \Rightarrow \frac{x+1}{8}$

$to\ the\ fourth\ customer \Rightarrow \frac{x-7}{16} + \frac{1}{2} \Rightarrow \frac{x+1}{16}$

$x - \frac{x+1}{2} - \frac{x+1}{4} - \frac{x+1}{8} - \frac{x+1}{16} = 15$

$$\frac{16x-8(x+1)-4(x+1)-2(x+1)-(x+1)}{16} = 15$$

$$\frac{16x-8x-8-4x-4-2x-2-x-1}{16} = 15$$

$$\frac{16x-15x-15}{16} = 15$$

$x - 15 = 15 \times 16$

$x - 15 = 240$

$x = 240 + 15 = 255$

Hence, the number of cakes he had in the beginning is 289.

Question 21

In the sequence $1, 1, 1, 2, 1, 3, 1, 4, 1, 5$, ..., the 2022^{nd} term is______

Solution: (1011)

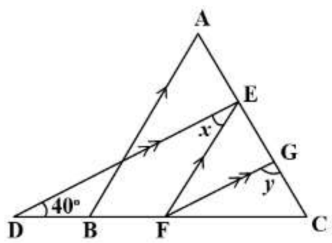
Before every natural number '1' is added in this sequence.

So, in the 10^{th} term we are getting '5'

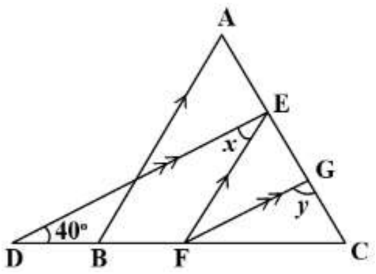
Similarly, For the 2022^{th} term we will be getting **1011.**

Question 22

In the adjoining figure, ABC is an equilateral triangle. AB and EF are parallel. DE and FG are parallel. $\angle BDE = 40^{\circ}$. Then $x + y$ (in degrees) is_____



Solution: $(100°)$



Given, $\Delta ABC\ is\ an\ equilateral\ triangle.$

$\therefore \angle A = \angle B = \angle C = 180°$

Let, $\angle DEF = \angle EFG = x\ (alternate\ interior\ angle)$

$\angle EDF = \angle GFC = 40°\ (Corresponding\ angle)$

In, $\Delta CFG,$

$y + 40° + 60° = 180°$

$y = 80°$

According to the exterior angle property,

$y = \angle EFG + \angle GEF$

$y = x + \angle GEF$

$\angle GEF = y - x$

$60° = 80° - x$

$x = 20°$

Then $x + y = 20° + 80° = 100°$

Question 23

A gardener has to plant a number of rose plants in straight rows. First he tried 5 in each row; then he successively tried 6, 8, 9 and 12 in each row but always had 1 plant left Then he tried 13 in a row and to his pleasant surprise, no plant was left out. The smallest number of plants he could have had is

Solution: (3601)

Number of plant $= LCM(5, 6, 8, 9, 12) \times K + 1$
$N = 360K + 1$

Also it is divisible by 13

$k = 1 \rightarrow N = 361\ \ It\ is\ not\ divisible\ by\ 13$

$k = 2 \quad \rightarrow N = 721\ It\ is\ not\ divisible\ by\ 13$

$k = 3 \rightarrow N = 1081\ It\ is\ not\ divisible\ by\ 13$

$k = 10 \qquad N = 3601 \Rightarrow It\ is\ divisible\ by\ 13$

The smallest number of plants he could have had is 3601.

Question 24

A, B run a race $1\ km$ long straight path. If A gives B $40\ m$ start then, A wins by 19 seconds. If A gives B 30 seconds start, then B wins by $40\ m$. If B normally would take t_1 seconds to run the total $1\ km$ length and A normally would take t_2 seconds to run the total $1\ km$ length, then $t_1 - t_2$ (in seconds) is

Solution: (25)

According to the given situation,

Let a and b be the speeds of the A and B respectively,

$19 = (1000 - 40)/(b - t_2)$

$$19 = \frac{960}{1000}t_1 - t_2$$

$$or\ t_2 = \frac{960}{1000}t_1 - 19\text{.......}\ (i)$$

$and,\ 30 = t_1 - (1000 - 40)/a$

$$30 = t_1 - \frac{960}{1000}t_2 or\ t_1 = 30 + \frac{960}{1000}t_2\text{..........}\ (ii)$$

Subtract (i) from (ii),

$$t_1 - t_2 =- \frac{96}{100}t_1 + \frac{96}{100}t_2 + 19 + 30$$

$$t_1 - t_2 = \frac{96}{100}\left[t_2 - t_1\right] + 49$$

$$t_1 - t_2 - \frac{96}{100}\left[t_2 - t_1\right] = 49$$

$$t_1 - t_2 + \frac{96}{100}\left(t_1 - t_2\right) = 49$$

$$\left[t_1 - t_2\right]\left[1 + \frac{96}{100}\right] = 49$$

$$t_1 - t_2 = 25$$

Question 25

David computed the value of 3^{19} as $11a2261467$. He found all the digits correctly except ' a '. The value of ' a ' is

Solution: (6)

3^{19} as $11a2261467$
$1 + 1 + a + 2 + 2 + 6 + 1 + 4 + 6 + 7 = 30 + a$

Value of $a = 6$

Question 26

The sum of eight consecutive natural numbers is 124 . The sum of the next 5 natural numbers will be

Solution: (110)
Sum of eight consecutive natural numbers $= 124$

$x + x + 1 + x + 2 + x + 3 + x + 4 + x + 5 + x + 6 + x + 7 = 124$

$8x + 28 = 124$

$8x = 124 - 28$

$8x = 96$

$x = \frac{96}{8}$

$x = 12$

Sum of next five natural numbers = $x + 8 + x + 9 + x + 10 + x + 11 + x + 12$

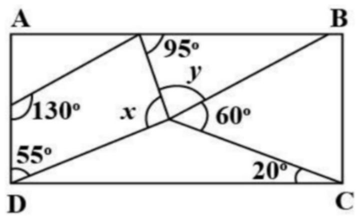
$= 12 + 8 + 12 + 9 + 12 + 10 + 12 + 11 + 12 + 12$

$= 20 + 21 + 22 + 23 + 24$

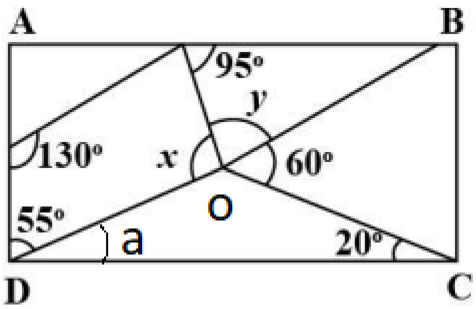
$= 110$

Question 27

In the adjoining figure, $ABCD$ is a rectangle. The value of $x + y$ (in degrees) is



Solution: (175°)



$ABCD$ is a rectangle.

$\angle D = 90^{\circ}$

$55^{\circ} + a = 90^{\circ}$

$a = 90^{\circ} - 55^{\circ}$

$a = 35^{\circ}$

In triangle DOC,.

$\angle ODC + \angle DOC + \angle OCD = 180^{\circ}$

$35^{\circ} + \angle O + 20 = 180$

$\angle O = 180 - 55 = 125^{\circ}$

We know the angle of a circle is 360° So,

$x + y + 60^{\circ} + 125^{\circ} = 360^{\circ}$

$x + y = 360^{\circ} - 185^{\circ}$

$x + y = 175^{\circ}$

Question 28

If $A = (625)^{-3/4}$ and $B = (78125)^{3/7}$, then the value of $A \times B$ is

Solution: (1)

$$A = (625)^{\frac{-3}{4}}, B = (78125)^{\frac{3}{7}}$$

$$A = \left(5^{4}\right)^{\frac{-3}{4}} = (5)^{-3} = \frac{1}{125}$$

$$B = \left(5^{7}\right)^{\frac{3}{7}} = 125$$

$A \times B \ = \frac{1}{125} \times 125 = 1$

Question 29

A room is $5\ m\ 44\ cm$ long and $3\ m\ 74\ cm$ broad. The side of the largest square-slabs which can be paved of this room (in $cm.$) is

Solution: $34\ cm$

The side of the square slab is the HCF of 544 & $374\ cm$ is 34

$544 = 2\times2\times2\times2\times2\times17$

$374 = 2\times11\times17$

In both Common factor is 2×17

The side of the largest square-slabs $= 34\ cm$

Question 30

A company sells umbrellas in two different sizes, large and small. This year it sold 200 umbrellas, of which one-fourth were large. The sale of large umbrellas produced one-third of the company's income. If a: b is the ratio of the price of a larger umbrella to the price of a smaller umbrella, then ab^2 is

Solution: (12)

Total umbrellas = 200

Large Umbrellas = 50

Small Umbrellas = 150

Let x = total income

Income for large and small umbrellas = $\frac{x}{3} : \frac{2x}{3}$

for 1 large umbrella and 1 small umbrella $= \frac{x}{3\times50} : \frac{2x}{3\times150}$

$= \frac{x}{3} : \frac{2x}{3\times3}$

$= 1 : \frac{2}{3}$

$a: b = 3: 2$

$ab^2 = 3\times2^2 = 12$

***Regd. Office**:* Aakash Tower, 8, Pusa Road, New Delhi-110005, Ph.011-47623456

ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
AMTI – NMTC - 2023 Jan. – JUNIOR – FINAL

Instructions:

1. Answer all questions. Each question carries 10 marks.
2. Elegant and innovative solutions will get extra marks.
3. Diagrams and justification should be given wherever necessary.
4. Before answering, fill in the FACE SLIP completely.
5. Your 'rough work' should be in the answer sheet itself.
6. The maximum time allowed is THREE hours.

1. $x,\ y,\ z$ are positive reals and $(x + y + z)^3 = 32\,xyz$. Find the numerical limits between which the expression $\frac{x^4+y^4+z^4}{(x+y+z)^4}$ lies?

Sol. $\frac{x^4+y^4+z^4}{(x+y+z)^4} = \frac{x^4}{(x+y+z)^4} + \frac{y^4}{(x+y+z)^4} + \frac{z^4}{(x+y+z)^4}$

$$= \left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4$$

Now, apply $AM \geq GM$ for numbers $\left(\frac{x}{x+y+z}\right)^4, \left(\frac{y}{x+y+z}\right)^4, \left(\frac{z}{x+y+z}\right)^4$ we get

$$\frac{\left(\frac{x}{x+y+z}\right)^4+\left(\frac{y}{x+y+z}\right)^4+\left(\frac{z}{x+y+z}\right)^4}{3} \geq \sqrt[3]{\left(\frac{xyz}{(x+y+z)^3}\right)^4}$$

We know that $(x + y + z)^3 = 32\,xyz$

$$\left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4 \geq 3\sqrt[3]{\left(\frac{1}{32}\right)^4}$$

$$\left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4 \geq 3.\left(\frac{1}{32}\right)^{\frac{4}{3}}$$

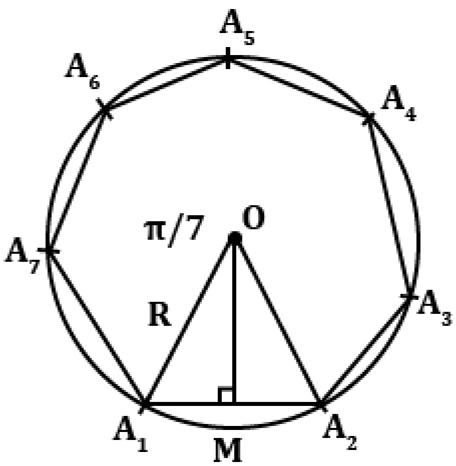
Or

$$\left(\frac{x}{x+y+z}\right)^4 + \left(\frac{y}{x+y+z}\right)^4 + \left(\frac{z}{x+y+z}\right)^4 \geq \frac{3}{(2)^{\frac{20}{3}}}$$

2. $A_1A_2A_3A_4A_5A_6A_7$ is a regular heptagon. Prove that

$$\frac{A_1A_4^{\,3}}{A_1A_2^{\,3}} - \frac{A_1A_7+2A_1A_6}{A_1A_5-A_1A_3} = 1$$

Sol. B



$A_1A_2 = A_1A_7$

$A_1A_3 = A_1A_6$

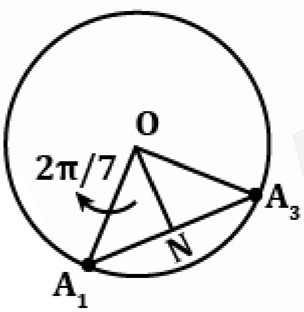
$A_1A_4 = A_1A_5$

Let radius of circumcircle = R

Centre = O

$\frac{A_1M}{R} = sin\,\frac{\pi}{7} \Rightarrow A_1M = R\,sin\frac{\pi}{7}$ $\qquad \angle A_iOA_2 = \frac{2\pi}{7}$

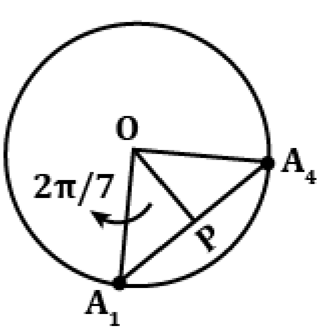
$\Rightarrow A_1A_2 = A_1A_7 = 2\,R\,sin/\pi\,\frac{\pi}{7}$ ………………(1)



$\angle A_1OA_3 = \frac{4\pi}{7}$

$A_1N = R\,sin\,\frac{2\pi}{7}$

$\Rightarrow A_1A_3 = 2d\,sin\,\frac{3\pi}{7} = A_1A_6$ ……………(2)



$\angle A_1OA_4 = \frac{4\pi}{7}$

$A_1P = R\,sin\,\frac{3\pi}{7}$

$\Rightarrow A_1A_4 = 2\,R\,sin\,\frac{3\pi}{7} = A_1A_5$ ……………(3)

Now To prove, $\frac{(A_1A_4)^3}{(A_1A_2)^3} - \frac{A_1A_1+2\,A_1A_6}{A_1A_5-A_1A_3} = 1$

To prove , $\frac{(A_1A_4)^3}{(A_1A_2)^3} = 1 + \frac{A_1A_7+2\,A_1A_6}{A_1A_5-A_1A_3}$

To prove $\left(\frac{A_1A_4}{A_1A_2}\right)^3 = 1 + \frac{A_1A_1+2\,A_1A_3}{A_1A_4-A_1A_3}$

To prove $\left(\frac{A_1A_4}{A_1A_2}\right)^3 = \frac{A_1A_2+A_1A_3+A_1A_4}{A_1A_4-A_1A_3}$

I.e., $\left(\frac{sin\,\frac{3\pi}{7}}{sin\,\frac{\pi}{7}}\right)^3 = \frac{sin\,\frac{\pi}{7}+sin\,\frac{2\pi}{7}+sin\,\frac{3\pi}{7}}{sin\,\frac{3\pi}{7}-sin\,\frac{2\pi}{7}}$

R.H.S = $\frac{sin\,\frac{\pi}{7}+sin\,\frac{2\pi}{7}+sin\,\frac{3\pi}{7}}{sin\,\frac{3\pi}{7}-sin\,\frac{2\pi}{7}}$

$= \frac{sin\,\frac{3\pi}{14}\,sin\,\frac{2\pi}{7}}{2sin\,\frac{\pi}{14}sin\,\frac{2\pi}{14}cos\,\frac{5\pi}{14}} = \frac{sin\,\frac{3\pi}{14}\,sin\,\frac{2\pi}{7}2cos\frac{\pi}{14}cos\frac{\pi}{14}}{2sin\,\frac{\pi}{14}sin\,\frac{\pi}{14}sin\,\frac{\pi}{7}2cos\frac{\pi}{14}cos\frac{\pi}{14}}$

$=\frac{sin\,\frac{3\pi}{14}\,sin\,\frac{2\pi}{7}2cos\frac{\pi}{14}cos\frac{\pi}{14}}{sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}} = \frac{2sin\,\frac{3\pi}{14}\,sin\,\frac{4\pi}{14}cos\frac{\pi}{14}cos\frac{\pi}{14}}{sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}}$

$= \frac{cos\,\frac{\pi}{14}\,cos\frac{\pi}{14}cos\frac{\pi}{14}}{sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}} = \frac{sin\,\frac{3\pi}{7}\,sin\frac{3\pi}{7}sin\frac{3\pi}{7}}{sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}sin\,\frac{\pi}{7}}$ = L.H.S.

3. $ABCD$ is a square whose side is $1\,unit$. Let n be an arbitrary natural number. A figure is drawn inside the square consisting of only line segments, having a total length greater than $2n$. (This figure can have many pieces of single line segments intersecting or non-intersecting). Prove that for some straight-line L which is parallel to a side of the square must cross the figure at least $(n+1)$ times.

Sol. Ambiguity in Question.

4. m is a natural number. If $(2m+1)$ and $(3m+1)$ are perfect squares, then prove that m is divisible by 40.

Sol. As 'm' is a natural number, to prove its divisibility by 40 , we prove that 'm' is divisible be 8 & 5.

Let $2m + 1 = k^2$……………..(i)

$3m + 1 = l^2$……………..(ii)

As $2m + 1$ is odd, $\therefore k^2$ is odd and thus k is odd.

So, let $k = 2n + 1 \Rightarrow k^2 = 4n^2 + 4n + 1$……………(iii)

$\Rightarrow 2m + 1 = 4n^2 + 4n + 1$

$\Rightarrow m = 2(n^2 + n)$

$\Rightarrow m$ = an even number

If 'm' is even, $3m + 1$ is odd.

$\Rightarrow l^2$ is odd & thus l is odd.

Let $l = 2p + 1 \Rightarrow l^2 = (2p + 1)^2$……..(iv)

Now, subtracting (ii) & (i),

$m = l^2 - k^2$

$= (2p + 1)^2 - (2n + 1)^2$………(v)

We know that, squares of two odd numbers are always divisible by 8

$\therefore$ m is divisible by 8 ……….(vi)

Also, from (i) & (ii)

$3k^2 - 2l^2 = 1$……….(vii)

As squares of odd numbers ends with $1, 5$ or 9

$\Rightarrow$ $3k^2$ ends with $3, 5$ or 7 & $2l^2$ ends with $2, 0, 8$

$\therefore$ k^2ends with 1 & l^2 ends with 1

$\Rightarrow$ $m = l^2 - k^2$ (whose unit digit is zero)

$\therefore m$ is divisible by 5 …..(viii)

From (vii) & (viii), m is divisible by 40.

5. Given 69 distinct positive integers not exceeding 100, prove that one can choose four of them a, b, c, d such that $a < b < c$ and $a + b + c = d$. Is this statement true for 68?

Sol. $a + b + c = d \Rightarrow c + a = d - b$

Let $a_i + a_1 = f_i$ $\forall$ $i \in \{3, 4, ..., 69\}$

& $a_i + a_2 = g_i$ $\forall$ $i \in \{3, 4, ..., 69\}$

Now

$a_3 + a_1 < a_4 + a_1 < .... < a_{69} + a_1 \leq 132$

And

$1 \leq a_3 - a_2 < a_4 - a_2 < .... < a_{69} - a_2$

$\Rightarrow f_3 < f_4 < .... < f_{69} \leq 132$

& $1 \leq g_3 + g_4 < .... < g_{69}$

Now total counting of f_i & g_i are 134 which lies between 1 to 132

Because all f_i's are distinct and all g_i's are distinct, hence at least one f_i and g_i must be same

$\Rightarrow a_i + a_1 = a_j - a_2$

$\Rightarrow a_1 + a_2 + a_i = a_j$

$\Rightarrow$ First part is proved

This statement is not true for 68 e.g. let set of 68 numbers is $\{33, 34, ..., 100\}$

In this set sum of least 3 number is greater than 100 $\{because\ 33 + 34 + 35 = 102 > 100\}$

6. m, n are integers such that $n^2(m^2 + 1) + m^2(n^2 + 16) = 448$. Find all possible ordered pairs (m, n).

Sol. $m^2n^2 + n^2 + m^2n^2 + 16m^2 = 448$

$\Rightarrow 2m^2n^2 + n^2 + 16m^2 = 448$

$\Rightarrow m^2n^2 + \frac{n^2}{2} + 8m^2 = 224$

$\Rightarrow (m^2 + \frac{1}{2})(n^2 + 8) = 228$

$\Rightarrow (2m^2 + 1)(n^2 + 8) = 456$

$\Rightarrow (2m^2 + 1)(n^2 + 8) = 3 \times 152$ or 19×24

Odd even

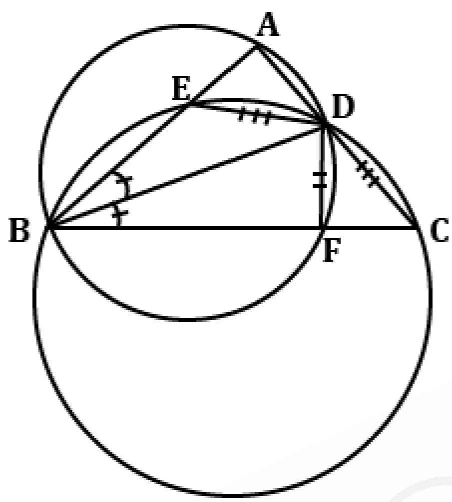
$\Rightarrow m^2 = 1, n^2 = 144$ or $m^2 = 9, n^2 = 16$

$\Rightarrow (m, n) = (1, 12), (1, -12), (-1, 12), (-1, -12)$

$(3, 4), (3, -4), (-3, 4), (-3, -4)$

7. BD is the bisector of $\angle ABC$ of triangle ABC. The circumcircles of triangle BCD and triangle ABD cut AB and BC at E and F respectively. Show that $AE = CF$.

Sol. Construction: Join DE and DF



Proof : $arc\ AD = arc\ DF$ for circumcircle of ΔABD

$\{\because \angle ABD = \angle DBF\}$

$\Rightarrow AD = DF$ ....(1)

Similarly $arc\ ED = arc\ DC$

For circumcircle of ΔBDC

$\Rightarrow ED = DC$ ....(2)

Now $\angle EDC = 180° - \angle ABC$

Similarly $\angle ADF = 180° - \angle ABC$

$\Rightarrow \angle EDC = \angle ADF$

$\Rightarrow \angle EDC - \angle EDF = \angle ADF - \angle EDF$

$\Rightarrow \angle FDC = \angle ADE$ ....(3)

Using (1),(2) & (3) $\Delta ADE \cong \Delta FDC$

$\Rightarrow AE = FC$ (using $CPCT$)

8. a is a two-digit number. b is a three-digit number. a increased by b percent is equal to b decreased by a percent. Find all possible ordered pairs (a, b).

Sol. $a + \frac{ab}{100} = b - \frac{ab}{100}$

$\Rightarrow \frac{ab}{50} + a - b = 0$

$\Rightarrow ab + 50a - 50b = 0$

$\Rightarrow (a - 50)(b + 50) = -2500$

$= -4 \times 625$ or

$= -5 \times 500$ or

$= -10 \times 250$ or

$\Rightarrow (a, b) = (46, 575)$ or $(45, 450)$ or $(40, 200)$