M1

Time: 3 Hours

Number of Questions: 30 **Max Marks: 100**

INSTRUCTIONS

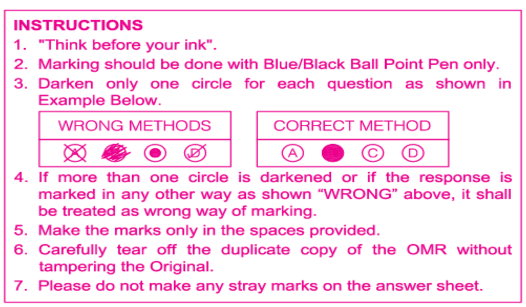
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

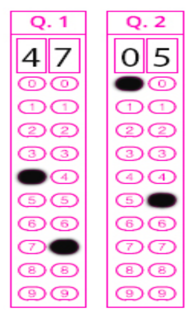
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer. .

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.

4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.





6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 and 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

Note:

1. $\mathbb{N}$ denotes the set of all natural numbers, $1, 2, 3, \ldots$.
2. For a positive real number x, $\sqrt{x}$ denotes the positive square root of x. For example, $\sqrt{4} = +2$.
3. Unless otherwise specified, all numbers are written in base 10.

Questions

1. Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of integers in the set $X_n = \{\sqrt{4n+1}, \sqrt{4n+2}, \ldots, \sqrt{4n+1000}\}$. Let

$$a = \max\{M_n : 1 \leq n \leq 1000\}, \text{ and } b = \min\{M_n : 1 \leq n \leq 1000\}.$$

Find $a - b$.

2. Find the number of elements in the set

$$\{(a, b) \in \mathbb{N} : 2 \leq a, b \leq 2023, \log_a(b) + 6 \log_b(a) = 5\}$$

3. Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}.$$

Find the smallest possible value of β.

4. Let x, y be positive integers such that

$$x^4 = (x-1)(y^3 - 23) - 1.$$

Find the maximum possible value of $x + y$.

SPACE FOR ROUGH WORK

5. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle GYZ.

6. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

8. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.

SPACE FOR ROUGH WORK

9. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

10. The sequence $\langle a_n \rangle_{n\geq 0}$ is defined by $a_0 = 1, a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

11. A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

SPACE FOR ROUGH WORK

13. The ex-radii of a triangle are $10\frac{1}{2}, 12$ and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$.

14. Let ABC be a triangle in the xy plane, where B is at the origin $(0,0)$. Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$ and AB be produced to F such that $AB : BF = 1 : 3$. Let $G(32, 24)$ be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

15. Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcentre of triangle MAN, and P be the circumcentre of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m + n$.

16. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \le i < j < k \le 6$, has at least one red side, find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

17. Consider the set

$$\mathcal{S} = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\}$$

where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of $\mathcal{S}$, find the largest integer less than or equal to D.

18. Let $\mathcal{P}$ be a convex polygon with 50 vertices. A set $\mathcal{F}$ of diagonals of $\mathcal{P}$ is said to be *minimally friendly* if any diagonal $d \in \mathcal{F}$ intersects at most one other diagonal in $\mathcal{F}$ at a point interior to $\mathcal{P}$. Find the largest possible number of elements in a minimally friendly set $\mathcal{F}$.

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set

$$A = \{n \in \mathbb{N} : P(n) \text{ is non-zero, square free and } S(n) \text{ is a proper divisor of } P(n)\}.$$

Find the maximum possible number of digits of the numbers in A.

SPACE FOR ROUGH WORK

20. For any finite non empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$$\max(A) \times |B| = 12; \text{ and}$$
$$|A| \times \max(B) = 11$$

and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$.

21. For $n \in \mathbb{N}$, consider non-negative integer-valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

22. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$.

24. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set $\{5, 6, 7, 8, 9, 10\}$.

25. Find the least positive integer n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon.

SPACE FOR ROUGH WORK

26. In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations $1, 2, 2^2, 2^3, \ldots$ Bens. The rules of the Government of Binary stipulate that one can not use more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government.

27. A quadruple (a, b, c, d) of distinct integers is said to be *balanced* if $a + c = b + d$. Let $\mathcal{S}$ be any set of quadruples (a, b, c, d) where $1 \leq a < b < d < c \leq 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in $\mathcal{S}$.

28. On each side of an equilateral triangle with side length n units, where n is an integer, $1 \leq n \leq 100$, consider $n - 1$ points that divide the side into n equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be *adjacent* if the distance between them is 1 unit. A *move* consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.

SPACE FOR ROUGH WORK

29. A positive integer $n > 1$ is called *beautiful* if n can be written in one and only one way as $n = a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k$ for some positive integers $a_1, a_2, \ldots, a_k$, where $k > 1$ and $a_1 \geq a_2 \geq \cdots \geq a_k$. (For example 6 is beautiful since $6 = 3 \cdot 2 \cdot 1 = 3 + 2 + 1$, and this is unique. But 8 is not beautiful since $8 = 4 + 2 + 1 + 1 = 4 \cdot 2 \cdot 1 \cdot 1$ as well as $8 = 2 + 2 + 2 + 1 + 1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$, so uniqueness is lost.) Find the largest beautiful number less than 100.

30. Let $d(m)$ denote the number of positive integer divisors of a positive integer m. If r is the number of integers $n \leq 2023$ for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.

SPACE FOR ROUGH WORK

Answers

QNo	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Answer	22	54	23	07	10	16	48	59	17	51	14	18	58	40	03

QNo	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Answer	94	66	71	92	43	15	77	18	31	28	19	91	67	95	18

M1

1. A triangle ABC with $AC = 20$ is inscribed in a circle ω. A tangent t to ω is drawn through B. The distance of t from A is 25 and that from C is 16. If S denotes the area of the triangle ABC, find the largest integer not exceeding $S/20$.

2. In a parallelogram $ABCD$, a point P on the segment AB is taken such that $\dfrac{AP}{AB} = \dfrac{61}{2022}$ and a point Q on the segment AD is taken such that $\dfrac{AQ}{AD} = \dfrac{61}{2065}$. If PQ intersects AC at T, find $\dfrac{AC}{AT}$ to the nearest integer.

3. In a trapezoid $ABCD$, the internal bisector of angle A intersects the base BC (or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and side BE at the point P. Find the angle DAE in degrees, if $AB : MP = 2$.

4. Starting with a positive integer M written on the board, Alice plays the following game: in each move, if x is the number on the board, she replaces it with $3x + 2$. Similarly, starting with a positive integer N written on the board, Bob plays the following game: in each move, if x is the number on the board, he replaces it with $2x + 27$. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of $M + N$.

5. Let m be the smallest positive integer such that $m^2 + (m+1)^2 + \cdots + (m+10)^2$ is the square of a positive integer n. Find $m + n$.

6. Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

7. Find the number of ordered pairs (a, b) such that $a, b \in \{10, 11, \cdots, 29, 30\}$ and
$$\text{GCD}(a, b) + \text{LCM}(a, b) = a + b.$$

8. Suppose the prime numbers p and q satisfy $q^2 + 3p = 197p^2 + q$. Write $\dfrac{q}{p}$ as $l + \dfrac{m}{n}$, where l, m, n are positive integers, $m < n$ and $\text{GCD}(m, n) = 1$. Find the maximum value of $l + m + n$.

9. Two sides of an integer sided triangle have lengths 18 and x where $x < 100$. If there are exactly 35 possible integer values y such that $18, x, y$ are the sides of a non-degenerate triangle, find the number of possible integer values x can have.

10. Consider the 10-digit number $M = 9876543210$. We obtain a new 10-digit number from M according to the following rule: we can choose one or more disjoint pairs of adjacent digits in M and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from $M = 9\underline{87}6\underline{54}3210$, by interchanging the 2 underlined pairs, and keeping the others in their places, we get $M_1 = 9\overline{78}6\overline{45}3210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from M.

11. Let AB be a diameter of a circle ω and let C be a point on ω, different from A and B. The perpendicular from C intersects AB at D and ω at $E(\neq C)$. The circle with centre at C and radius CD intersects ω at P and Q. If the perimeter of the triangle PEQ is 24, find the length of the side PQ.

12. Given ΔABC with $\angle B = 60^\circ$ and $\angle C = 30^\circ$, let P, Q, R be points on sides BA, AC, CB respectively such that $BPQR$ is an isosceles trapezium with $PQ \parallel BR$ and $BP = QR$. Find the minimum possible value of $\dfrac{2[ABC]}{[BPQR]}$ where $[S]$ denotes the area of any polygon S.

13. Let ABC be a triangle and let D be a point on the segment BC such that $AD = BC$. Suppose $\angle CAD = x^\circ$, $\angle ABC = y^\circ$ and $\angle ACB = z^\circ$ and x, y, z are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees.

14. Let x, y, z be complex numbers such that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$$
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 64$$
$$\frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} = 488$$

If $\dfrac{x}{yz} + \dfrac{y}{zx} + \dfrac{z}{xy} = \dfrac{m}{n}$ where m, n are positive integers with $\text{GCD}(m,n) = 1$, find $m + n$.

15. Let x, y be real numbers such that $xy = 1$. Let T and t be the largest and the smallest values of the expression

$$\frac{(x+y)^2 - (x-y) - 2}{(x+y)^2 + (x-y) - 2}.$$

If $T+t$ can be expressed in the form $\dfrac{m}{n}$ where m, n are nonzero integers with $\text{GCD}(m,n) = 1$, find the value of $m + n$.

16. Let a, b, c be reals satisfying

$$3ab + 2 = 6b, \qquad 3bc + 2 = 5c, \qquad 3ca + 2 = 4a.$$

Let $\mathbb{Q}$ denote the set of all rational numbers. Given that the product abc can take two values $\dfrac{r}{s} \in \mathbb{Q}$ and $\dfrac{t}{u} \in \mathbb{Q}$, in lowest form, find $r + s + t + u$.

17. For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of n and $f(n) = n - g(n)$. For example, $g(10) = 5, f(10) = 5$ and $g(13) = 1, f(13) = 12$. Let N be the smallest positive integer such that $f(f(f(N))) = 97$. Find the largest integer not exceeding $\sqrt{N}$.

18. Let m, n be natural numbers such that

$$m + 3n - 5 = 2\,\text{LCM}(m,n) - 11\,\text{GCD}(m,n).$$

Find the maximum possible value of $m + n$.

19. Consider a string of n 1's. We wish to place some $+$ signs in between so that the sum is 1000. For instance, if $n = 190$, one may put $+$ signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place $+$ signs so as to get the sum 1000, then find the sum of the digits of a.

20. For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \ldots, p_n)$ of $\{1, 2, \ldots, n\}$, we say p_l is a *landmark point* if $2 \leq l \leq n-1$ and $(p_{l-1} - p_l)(p_{l+1} - p_l) > 0$. For example, for $n = 7$, the permutation $(2, 7, 6, 4, 5, 1, 3)$ has four landmark points: $p_2 = 7, p_4 = 4, p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let $L(n)$ denote the number of permutations of $\{1, 2, \ldots, n\}$ with exactly one landmark point. Find the maximum $n \geq 3$ for which $L(n)$ is a perfect square.

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If N is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of N.

22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called *friendly* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is friendly. Let F_n denote the number of friendly binary sequences with n terms. Find the smallest positive integer $n \geq 2$ such that $F_n > 100$.

23. In a triangle ABC, the median AD divides $\angle BAC$ in the ratio $1:2$. Extend AD to E such that EB is perpendicular AB. Given that $BE = 3$, $BA = 4$, find the integer nearest to BC^2.

24. Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where a, b are positive integers less than 100, find $a + b$.

Answers

QNo	1	2	3	4	5	6	7	8	9	10	11	12
Answer	10	67	60	10	95	43	35	32	82	88	08	03

QNo	13	14	15	16	17	18	19	20	21	22	23	24
Answer	59	16	25	18	19	70	09	03	74	11	29	81

IOQM 2021-22 Part A

1. Three parallel lines L_1, L_2, L_3 are drawn in the plane such that the perpendicular distance between L_1 and L_2 is 3 and the perpendicular distance between L_2 and L_3 is also 3. A square $ABCD$ is constructed such that A lies on L_1, B lies on L_3 and C lies on L_2. Find the area of the square.

2. Ria writes down the numbers $1, 2, \ldots, 101$ in red and blue pens. The largest blue number is equal to the number of numbers written in blue and the smallest red number is equal to half the number of numbers written in red. How many numbers did Ria write with red pen?

3. Consider the set $\mathcal{T}$ of all triangles whose sides are distinct prime numbers which are also in arithmetic progression. Let $\triangle \in \mathcal{T}$ be the triangle with the least perimeter. If a° is the largest angle of $\triangle$ and if L is its perimeter, determine the value of $\frac{a}{L}$.

4. Consider the set of all 6-digit numbers consisting of only 3 digits, a, b, c, where a, b, c are distinct. Suppose the sum of all these numbers is 593999406. What is the largest remainder when the three digit number abc is divided by 100?

5. In parallelogram $ABCD$ the longer side is twice the shorter side. Let $XYZW$ be the quadrilateral formed by the internal bisectors of the angles of $ABCD$. If the area of $XYZW$ is 10, find the area of $ABCD$.

6. Let x, y, z be positive real numbers such that $x^2 + y^2 = 49$, $y^2 + yz + z^2 = 36$ and $x^2 + \sqrt{3}xz + z^2 = 25$. If the value of $2xy + \sqrt{3}yz + zx$ can be written as $p\sqrt{q}$ where p, q are integers and q is not divisible by square of any prime number, find $p + q$.

7. Find the number of maps $f : \{1, 2, 3\} \longrightarrow \{1, 2, 3, 4, 5\}$ such that $f(i) \le f(j)$ whenever $i < j$.

8. For any real number t, let $\lfloor t \rfloor$ denote the largest integer $\le t$. Suppose that N is the greatest integer such that

$$\left\lfloor \sqrt{\left\lfloor \sqrt{\left\lfloor \sqrt{N} \right\rfloor} \right\rfloor} \right\rfloor = 4$$

   Find the sum of digits of N.

9. Let $P_0 = (3, 1)$ and define $P_{n+1} = (x_n, y_n)$ for $n \ge 0$ by

$$x_{n+1} = -\frac{3x_n - y_n}{2}, \qquad y_{n+1} = -\frac{x_n + y_n}{2}$$

   Find the area of the quadrilateral formed by the points $P_{96}, P_{97}, P_{98}, P_{99}$.

10. Suppose that P is the polynomial of least degree with integer coefficients such that $P\left(\sqrt{7} + \sqrt{5}\right) = 2\left(\sqrt{7} - \sqrt{5}\right)$. Find $P(2)$.

11. In how many ways can four married couples sit in a merry-go-round with identical seats such that men and women occupy alternate seats and no husband seats next to his wife?

12. A 12×12 board is divided into 144 unit squares by drawing lines parallel to the sides. Two rooks placed on two unit squares are said to be non attacking if they are not in the same column or same row. Find the least number N such that if N rooks are placed on the unit squares, one rook per square, we can always find 7 rooks such that no two are attacking each other.

Code : **M01**

Roll Number: | | | | | | | | | | | | | | | | |

Time: 3 Hours

Number of Questions: 30 **Max Marks: 100**

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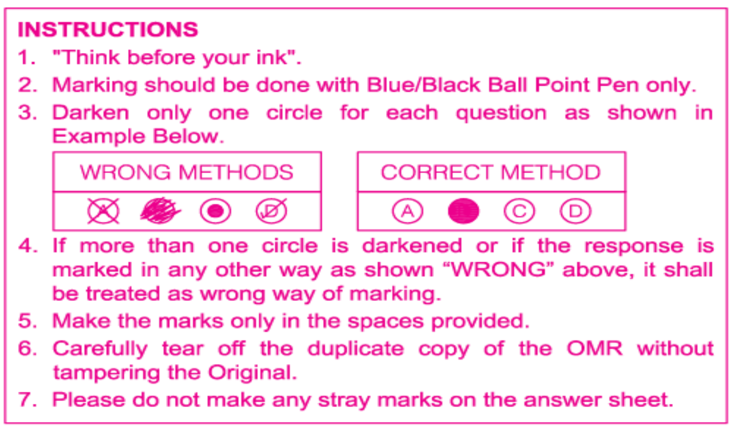

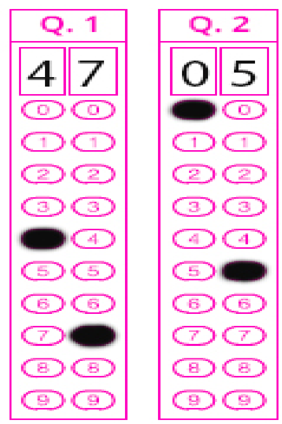

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1. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AB = 3CD$. Let E be the midpoint of the diagonal BD. If $[ABCD] = n \times [CDE]$, what is the value of n? (Here $[\Gamma]$ denotes the area of the geometrical figure Γ.)

2. A number N in base 10, is 503 in base b and 305 in base $b+2$. What is the product of the digits of N?

3. If $\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2} = 0.9999$ then determine the value of N.

4. Let $ABCD$ be a rectangle in which $AB + BC + CD = 20$ and $AE = 9$ where E is the mid-point of the side BC. Find the area of the rectangle.

5. Find the number of integer solutions to $\left||x| - 2020\right| < 5$.

6. What is the least positive integer by which $2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$ should be multiplied so that, the product is a perfect square ?

7. Let ABC be a triangle with $AB = AC$. Let D be a point on the segment BC such that $BD = 48\frac{1}{61}$ and $DC = 61$. Let E be a point on AD such that CE is perpendicular to AD and $DE = 11$. Find AE.

8. A 5-digit number (in base 10) has digits k, $k+1$, $k+2$, $3k$, $k+3$ in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.

SPACE FOR ROUGH WORK

9. Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The internal angle bisector of C intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively, such that $DM \parallel AC$ and $DN \parallel BC$. If $(MN)^2 = \frac{p}{q}$ where p and q are relatively prime positive integers then what is the sum of the digits of $|p - q|$?

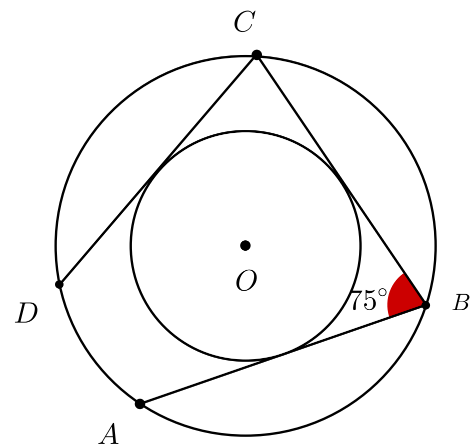
10. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores? (The median of a set of scores is the middlemost score when the data is arranged in increasing order. It is exactly the middle score when there are an odd number of scores and it is the average of the two middle scores when there are an even number of scores.)

11. Let $X = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and

$$S = \{(a, b) \in X \times X : x^2 + ax + b \text{ and } x^3 + bx + a \text{ have at least a common real zero}\}.$$

How many elements are there in S?

12. Given a pair of concentric circles, chords $AB, BC, CD, \ldots$ of the outer circle are drawn such that they all touch the inner circle. If $\angle ABC = 75°$, how many chords can be drawn before returning to the starting point?



SPACE FOR ROUGH WORK

13. Find the sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square.

14. The product $55 \times 60 \times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

15. Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the other. How many arrangements are possible?

16. The sides x and y of a scalene triangle satisfy $x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}$, where Δ is the area of the triangle. If $x = 60, y = 63$, what is the length of the largest side of the triangle?

SPACE FOR ROUGH WORK

17. How many two digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n.)

18. If

$$\sum_{k=1}^{40}\left(\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\right)=a+\frac{b}{c}$$

where $a, b, c \in \mathbb{N}, b < c, \gcd(b, c) = 1$, then what is the value of $a + b$?

19. Let $ABCD$ be a parallelogram . Let E and F be midpoints of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of a triangle among these four triangles?

20. A group of women working together at the same rate can build a wall in 45 hours. When the work started, all the women did not start working together. They joined the work over a period of time, one by one, at equal intervals. Once at work, each one stayed till the work was complete. If the first woman worked 5 times as many hours as the last woman, for how many hours did the first woman work?

21. A total fixed amount of N thousand rupees is given to three persons A, B, C, every year, each being given an amount proportional to her age. In the first year, A got half the total amount. When the sixth payment was made, A got six-seventh of the amount that she had in the first year; B got Rs 1000 less than that she had in the first year; and C got twice of that she had in the first year. Find N.

SPACE FOR ROUGH WORK

22. In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of $\angle ABC$, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle ACB$, respectively. If $PQ = 7$, $QR = 6$ and $RS = 8$, what is the area of triangle ABC?

23. The incircle Γ of a scalene triangle ABC touches BC at D, CA at E and AB at F. Let r_A be the radius of the circle inside ABC which is tangent to Γ and the sides AB and AC. Define r_B and r_C similarly. If $r_A = 16$, $r_B = 25$ and $r_C = 36$, determine the radius of Γ.

24. A light source at the point $(0, 16)$ in the coordinate plane casts light in all directions. A disc (a circle along with its interior) of radius 2 with center at $(6, 10)$ casts a shadow on the X axis. The length of the shadow can be written in the form $m\sqrt{n}$ where m, n are positive integers and n is square-free. Find $m + n$.

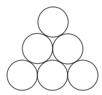
25. For a positive integer n, let $\langle n \rangle$ denote the perfect square integer closest to n. For example, $\langle 74 \rangle = 81, \langle 18 \rangle = 16$. If N is the smallest positive integer such that

$$\langle 91 \rangle \cdot \langle 120 \rangle \cdot \langle 143 \rangle \cdot \langle 180 \rangle \cdot \langle N \rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot N$$

find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

26. In the figure below, 4 of the 6 disks are to be colored black and 2 are to be colored white. Two colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same.



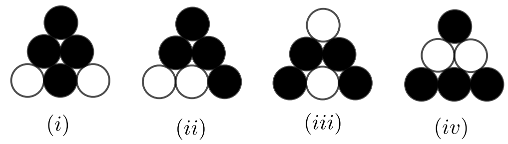


Fig 1.

There are only four such colorings for the given two colors, as shown in Figure 1. In how many ways can we color the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored blue with the given identification condition?

27. A bug travels in the coordinate plane moving only along the lines that are parallel to the x axis or y axis. Let $A = (-3, 2)$ and $B(3, -2)$. Consider all possible paths of the bug from A to B of length at most 14. How many points with integer coordinates lie on at least one of these paths?

28. A natural number n is said to be *good* if n is the sum of r consecutive positive integers, for some $r \geq 2$. Find the number of good numbers in the set $\{1, 2, \ldots, 100\}$.

29. Positive integers a, b, c satisfy $\dfrac{ab}{a-b} = c$. What is the largest possible value of $a+b+c$ not exceeding 99?

30. Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, $a+1$ divides $b-1$ and b divides a^2+a+2.

SPACE FOR ROUGH WORK

ROUGH WORK